

18. DIFFERENTIAL EQUATIONS AND THEIR FORMATION

DIFFERENTIAL EQUATION An equation containing an independent variable, a dependent variable and the derivatives of the dependent variable is called a differential equation.

Examples Each of the following equations is a differential equation:

$$\begin{array}{ll} \text{(i)} \quad \frac{dy}{dx} + 5y = e^x & \text{(ii)} \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \sin x \\ \text{(iii)} \quad \frac{dy}{dx} = \frac{x^3 - y^3}{xy^2 - x^2y} & \text{(iv)} \quad x^2dx + y^2dy = 0 \end{array}$$

ORDER OF A DIFFERENTIAL EQUATION The order of the highest-order derivative occurring in a differential equation is called the order of the differential equation.

DEGREE OF A DIFFERENTIAL EQUATION The power of the highest-order derivative occurring in a differential equation, after it is made free from radicals and fractions, is called the degree of the differential equation.

EXAMPLE 1 Write the order and the degree of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 3y = 0.$$

SOLUTION In the given equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

EXAMPLE 2 Write the order and the degree of the differential equation

$$x\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + y^2 = 0.$$

SOLUTION In the given equation, the highest-order derivative is $\frac{d^3y}{dx^3}$ and its power is 2.

$\therefore$ its order = 3 and degree = 2.

EXAMPLE 3 Write the order and the degree of the differential equation

$$y = x\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$



SOLUTION The given equation may be written as

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \left(y - x \frac{dy}{dx}\right) \\ \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 &= \left(y - x \frac{dy}{dx}\right)^2 \quad [\text{on squaring both sides}] \\ \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 &= y^2 + x^2 \left(\frac{dy}{dx}\right)^2 - 2xy \frac{dy}{dx} \\ \Rightarrow (1 - x^2) \left(\frac{dy}{dx}\right)^2 + 2xy \frac{dy}{dx} + (1 - y^2) &= 0.\end{aligned}$$

Clearly, it is a differential equation of order = 1 and degree = 2.

EXAMPLE 4 Write the order and degree of the differential equation

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} = k \left(\frac{d^2y}{dx^2}\right).$$

SOLUTION On squaring both sides, we get

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = k^2 \left(\frac{d^2y}{dx^2}\right)^2.$$

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ whose power is 2.

$\therefore$ its order = 2 and degree = 2.

EXAMPLE 5 Find the order and the degree of the differential equation

$$y = px + \sqrt{a^2p^2 + b^2}, \text{ where } p = \frac{dy}{dx}.$$

SOLUTION

$$\begin{aligned}y &= px + \sqrt{a^2p^2 + b^2} \\ \Rightarrow y - px &= \sqrt{a^2p^2 + b^2} \\ \Rightarrow (y - px)^2 &= a^2p^2 + b^2 \quad [\text{on squaring both sides}] \\ \Rightarrow y^2 + x^2p^2 - 2xyp &= a^2p^2 + b^2 \\ \Rightarrow (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) &= 0 \\ \Rightarrow (x^2 - a^2) \left(\frac{dy}{dx}\right)^2 - 2xy \cdot \left(\frac{dy}{dx}\right) + (y^2 - b^2) &= 0.\end{aligned}$$

Clearly, it is a differential equation of order 1 and degree 2.

AN IMPORTANT NOTE In case of differential equations involving one or more terms of the form $e^{(dy/dx)}$, $\log \left(\frac{dy}{dx}\right)$, $\sin \left(\frac{dy}{dx}\right)$, $\cos \left(\frac{dy}{dx}\right)$, etc., the degree is not defined.

However, the degree of the differential equation containing terms like e^x , e^y , $\log x$, $\log y$, $\tan x$, $\tan y$, etc., is defined as usual.



EXAMPLE 6 Find the order and degree (if any) of each of the differential equations given below:

- (i) $\frac{dy}{dx} - \tan x = 0$ (ii) $\left(\frac{dy}{dx}\right)^2 + y = e^x$
 (iii) $\frac{d^2y}{dx^2} = \sin 3x + \cos 3x$ (iv) $(y'')^2 + \cos y' = 0$
 (v) $y'' + 2y' + \sin y = 0$ (vi) $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$
 (vii) $y''' + y^2 + e^{y'} = 0$ (viii) $3\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 = \log x$

SOLUTION (i) The given equation is $\frac{dy}{dx} - \tan x = 0$.

In this equation, the highest-order derivative is $\frac{dy}{dx}$ whose power is 1.
 $\therefore$ its order = 1 and degree = 1.

(ii) The given equation is $\left(\frac{dy}{dx}\right)^2 + y = e^x$.

In this equation, the highest-order derivative is $\frac{dy}{dx}$ whose power is 2.
 $\therefore$ its order = 1 and degree = 2.

(iii) The given equation is $\frac{d^2y}{dx^2} = \sin 3x + \cos 3x$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.
 $\therefore$ its order = 2 and degree = 1.

(iv) The given equation is $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$, so its order is 2.

It has a term $\cos\left(\frac{dy}{dx}\right)$, so its degree is not defined.

(v) The given equation is $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \sin y = 0$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.
 $\therefore$ its order = 2 and degree = 1.



(vi) The given equation is $\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$.

In this equation, the highest-order derivative is $\frac{d^4y}{dx^4}$, so its order is 4.

It has a term $\sin\left(\frac{d^3y}{dx^3}\right)$, so its degree is not defined.

(vii) The given equation is $\frac{d^3y}{dx^3} + y^2 + e^{(dy/dx)} = 0$.

In this equation, the highest-order derivative is $\frac{d^3y}{dx^3}$, so its order is 3.

It has a term $e^{(dy/dx)}$, so its degree is not defined.

(viii) The given equation is $3\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 = \log x$.

In this equation, the highest-order derivative is $\frac{d^2y}{dx^2}$ and its power is 1.

$\therefore$ its order = 2 and degree = 1.

EXERCISE 18A

Write order and degree (if defined) of each of the following differential equations.

1. $\left(\frac{dy}{dx}\right)^4 + 3y\left(\frac{d^2y}{dx^2}\right) = 0$

[CBSE 2008C, '13C]

2. $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$

[CBSE 2013]

3. $\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{ds}{dt}\right)^3 + 4 = 0$

[CBSE 2013]

4. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y^5 = 0$

5. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$

6. $\frac{dy}{dx} + y = e^x$

7. $\frac{d^2y}{dx^2} + y^2 + e^{(dy/dx)} = 0$

8. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$

9. $\frac{d^4y}{dx^4} - \cos\left(\frac{d^3y}{dx^3}\right) = 0$

10. $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$

[CBSE 2010]



$$11. \left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$$

$$12. \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$13. x\left(\frac{dy}{dx}\right) + \frac{2}{\left(\frac{dy}{dx}\right)} + 9 = y^2$$

$$14. \sqrt{1 - \left(\frac{dy}{dx}\right)^2} = \left(a \frac{d^2y}{dx^2}\right)^{1/3}$$

$$15. \sqrt{1 - y^2} dx + \sqrt{1 - x^2} dy = 0$$

$$16. (y'')^3 + (y')^2 + \sin y' + 1 = 0$$

$$17. (3x + 5y)dy - 4x^2dx = 0$$

$$18. y = \frac{dy}{dx} + \frac{5}{\left(\frac{dy}{dx}\right)}$$

ANSWERS (EXERCISE 18A)

1. 2, 1 2. 2, 2 3. 2, 2 4. 3, 2 5. 2, 1 6. 1, 1 7. 2, not defined
 8. 1, not defined 9. 4, not defined 10. 2, 1 11. 1, 3 12. 3, 1 13. 1, 2
 14. 2, 2 15. 1, 1 16. 2, not defined 17. 1, 1 18. 1, 2

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18A)

13. The given differential equation on simplification becomes

$$x\left(\frac{dy}{dx}\right)^2 + 2 + 9\left(\frac{dy}{dx}\right) = y^2\left(\frac{dy}{dx}\right).$$

So, its order = 1 and degree = 2.

14. On squaring both sides, $\left\{1 - \left(\frac{dy}{dx}\right)^2\right\} = \left(a \frac{d^2y}{dx^2}\right)^{2/3}$ (i)

On cubing both sides of (i), we get

$$\left\{1 - \left(\frac{dy}{dx}\right)^2\right\}^3 = \left(a \frac{d^2y}{dx^2}\right)^2. \text{ So, its order} = 2 \text{ and degree} = 2.$$

15. $\frac{dy}{dx} = \frac{-\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$. So, its order = 1 and degree = 1.

17. $\frac{dy}{dx} = \frac{4x^2}{(3x + 5y)}$. So, its order = 1 and degree = 1.

SOLUTION OF A DIFFERENTIAL EQUATION A function of the form $y = f(x) + C$ which satisfies a given differential equation is called its solution.

GENERAL SOLUTION OF A DIFFERENTIAL EQUATION Suppose a differential equation of order n is being given. If its solution contains n arbitrary constants then it is called a general solution.



PARTICULAR SOLUTION OF A DIFFERENTIAL EQUATION Giving particular values to arbitrary constants in the general solution of a differential equation, we get its particular solutions.

SOLVED EXAMPLES

EXAMPLE 1 Verify that $y = A \cos x - B \sin x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0. \quad [\text{CBSE 2005C, '06}]$$

SOLUTION Given: $y = A \cos x - B \sin x$... (i)

$$\Rightarrow \frac{dy}{dx} = -A \sin x - B \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -A \cos x + B \sin x$$

$$= -(A \cos x - B \sin x) = -y \quad [\text{from (i)}]$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0.$$

Hence, $y = A \cos x - B \sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

EXAMPLE 2 Verify that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0. \quad [\text{CBSE 2003C}]$$

SOLUTION Given: $y = ae^{2x} + be^{-x}$... (i)

$$\Rightarrow \frac{dy}{dx} = 2ae^{2x} - be^{-x} \quad \dots \text{(ii)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \quad \dots \text{(iii)}$$

$$\therefore \left(\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y \right) = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x})$$

$$= 0 \quad [\text{using (i), (ii) and (iii)}].$$

Hence, $y = ae^{2x} + be^{-x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

EXAMPLE 3 Verify that $y = Ax + \frac{B}{x}$ is a solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0.$$

SOLUTION Given: $y = Ax + \frac{B}{x}$... (i)



$$\Rightarrow \frac{dy}{dx} = A - \frac{B}{x^2} \quad \dots \text{(ii)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2B}{x^3} \quad \dots \text{(iii)}$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (i), (ii) and (iii) respectively, we get

$$\begin{aligned} \left(x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y \right) &= x^2 \cdot \frac{2B}{x^3} + x \left(A - \frac{B}{x^2} \right) - \left(Ax + \frac{B}{x} \right) \\ &= \left(\frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} \right) = 0. \end{aligned}$$

Thus, $y = Ax + \frac{B}{x}$ satisfies $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0$.

Hence, $y = Ax + \frac{B}{x}$ is a solution of $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0$.

EXAMPLE 4 Verify that $y = a \cos(\log x) + b \sin(\log x)$ is a solution of the differential equation $x^2 + \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$. [CBSE 2007]

SOLUTION Given: $y = a \cos(\log x) + b \sin(\log x)$... (i)

$$\Rightarrow \frac{dy}{dx} = \frac{-a \sin(\log x)}{x} + \frac{b \cos(\log x)}{x} \quad [\text{on differentiating (i) w.r.t. } x]$$

$$\Rightarrow x \frac{dy}{dx} = -a \sin(\log x) + b \cos(\log x) \quad \dots \text{(ii)}$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-a \cos(\log x)}{x} - \frac{b \sin(\log x)}{x}$$

[on differentiating (ii) w.r.t. x]

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -a \cos(\log x) - b \sin(\log x)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0 \quad [\text{using (i)}].$$

Hence, $y = a \cos(\log x) + b \sin(\log x)$ is a solution of

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

EXAMPLE 5 Verify that $y = e^{m \sin^{-1} x}$ is a solution of the differential equation

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0.$$

SOLUTION Given: $y = e^{m \sin^{-1} x}$... (i)

$$\Rightarrow \frac{dy}{dx} = \frac{e^{m \sin^{-1} x}}{\sqrt{1 - x^2}} \cdot m \quad [\text{on differentiating (i)}]$$



$$\Rightarrow \sqrt{1-x^2} \left(\frac{dy}{dx} \right) = my \quad \dots \text{(ii)} \quad [\because e^{m \sin^{-1} x} = y]$$

$$\Rightarrow (1-x^2) \left(\frac{dy}{dx} \right)^2 = m^2 y^2 \quad \dots \text{(iii)} \quad [\text{on squaring both sides of (ii)}]$$

$$\Rightarrow (1-x^2) 2 \frac{dy}{dx} \cdot \left(\frac{d^2 y}{dx^2} \right) - 2x \left(\frac{dy}{dx} \right)^2 = 2m^2 y \frac{dy}{dx}$$

[on differentiating both sides of (iii)]

$$\Rightarrow 2 \left(\frac{dy}{dx} \right) \cdot \left\{ (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - m^2 y \right\} = 0$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0.$$

Hence, $y = e^{m \sin^{-1} x}$ is a solution of the differential equation

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - m^2 y = 0.$$

EXAMPLE 6 Verify that $v = \frac{A}{r} + B$ is a solution of the differential equation

$$\frac{d^2 v}{dr^2} + \frac{2}{r} \cdot \frac{dv}{dr} = 0.$$

SOLUTION Since the given relation contains two arbitrary constants, we differentiate it two times w.r.t. r , and eliminate A and B .

$$v = \frac{A}{r} + B \Rightarrow \frac{dv}{dr} = \frac{-A}{r^2} \quad \dots \text{(i)}$$

$$\Rightarrow \frac{d^2 v}{dr^2} = \frac{2A}{r^3} \quad \dots \text{(ii)}$$

On dividing (ii) by (i), we get

$$\frac{(d^2 v/dr^2)}{(dv/dr)} = \left\{ \frac{2A}{r^3} \times \frac{r^2}{(-A)} \right\} = \frac{-2}{r}$$

$$\Rightarrow \frac{d^2 v}{dr^2} = \frac{-2}{r} \cdot \frac{dv}{dr}$$

$$\Rightarrow \frac{d^2 v}{dr^2} + \frac{2}{r} \cdot \frac{dv}{dr} = 0.$$

Hence, $v = \frac{A}{r} + B$ is a solution of the differential equation

$$\frac{d^2 v}{dr^2} + \frac{2}{r} \cdot \frac{dv}{dr} = 0.$$

EXAMPLE 7 Prove that $(x^2 - y^2) = c(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where c is a parameter.



SOLUTION We have

$$(x^2 - y^2) = c(x^2 + y^2)^2. \quad \dots (i)$$

On differentiating (i) w.r.t. x , we get

$$\begin{aligned} 2x - 2y \frac{dy}{dx} &= 2c(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) \\ \Rightarrow \left(x - y \frac{dy}{dx} \right) &= 2c(x^2 + y^2) \left(x + y \frac{dy}{dx} \right) \quad \dots (ii) \\ \Rightarrow \left(x - y \frac{dy}{dx} \right) &= \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} (x^2 + y^2) \left(x + y \frac{dy}{dx} \right) \end{aligned}$$

[putting the value of c from (i) in (ii)]

$$\begin{aligned} \Rightarrow (x^2 + y^2) \left(x - y \frac{dy}{dx} \right) &= 2(x^2 - y^2) \left(x + y \frac{dy}{dx} \right) \\ \Rightarrow \{x(x^2 + y^2) - 2x(x^2 - y^2)\} &= \{2y(x^2 - y^2) + y(x^2 + y^2)\} \frac{dy}{dx} \\ \Rightarrow (3xy^2 - x^3) &= (3x^2y - y^3) \frac{dy}{dx} \\ \Rightarrow (x^3 - 3xy^2) dx &= (y^3 - 3x^2y) dy, \end{aligned}$$

which is the required differential equation.

EXAMPLE 8 Prove that $xy = ae^x + be^{-x} + x^2$ is the general solution of the differential equation $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 = 0$.

SOLUTION We have

$$xy = ae^x + be^{-x} + x^2. \quad \dots (i)$$

On differentiating (i) w.r.t. x , we get

$$x \frac{dy}{dx} + y = ae^x - be^{-x} + 2x. \quad \dots (ii)$$

On differentiating (ii) w.r.t. x , we get

$$\begin{aligned} x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} &= ae^x + be^{-x} + 2 \\ \Rightarrow x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} &= xy - x^2 + 2 \quad [\because \text{from (i), } (ae^x + be^{-x}) = (xy - x^2)] \\ \Rightarrow x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy + x^2 - 2 &= 0, \end{aligned}$$

which is the required differential equation.

EXAMPLE 9 Verify that $y = Ae^{ax} \cos bx + Be^{ax} \sin bx$, where A and B are arbitrary constants, is the general solution of the differential equation $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$.

SOLUTION We have

$$y = Ae^{ax} \cos bx + Be^{ax} \sin bx. \quad \dots (i)$$



Differentiating (i) on both sides w.r.t. x , we get

$$\frac{dy}{dx} = A \cdot \{e^{ax}(-b \sin bx) + ae^{ax} \cos bx\} + B \cdot \{e^{ax}(b \cos bx) + ae^{ax} \sin bx\}$$

$$\Rightarrow \frac{dy}{dx} = a\{Ae^{ax} \cos bx + Be^{ax} \sin bx\} + b\{-Ae^{ax} \sin bx + Be^{ax} \cos bx\}$$

$$\Rightarrow \frac{dy}{dx} = ay + be^{ax}\{B \cos bx - A \sin bx\} \quad \dots \text{(ii)} \quad [\text{using (i)}]$$

Differentiating (ii) on both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = a \frac{dy}{dx} + b[e^{ax}(-bB \sin bx - bA \cos bx) + (B \cos bx - A \sin bx)ae^{ax}]$$

$$\Rightarrow \frac{d^2y}{dx^2} = a \frac{dy}{dx} - b^2\{Ae^{ax} \cos bx + Be^{ax} \sin bx\} + a\{be^{ax}(B \cos bx - A \sin bx)\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = a \frac{dy}{dx} - b^2y + a\left(\frac{dy}{dx} - ay\right) \quad [\text{using (i) and (ii)}]$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0,$$

which is the required differential equation.

EXERCISE 18B

1. Verify that $x^2 = 2y^2 \log y$ is a solution of the differential equation $(x^2 + y^2) \frac{dy}{dx} - xy = 0$.
2. Verify that $y = e^x \cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$. [CBSE 2006]
3. Verify that $y = e^{m \cos^{-1} x}$ is a solution of the differential equation $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2y = 0$.
4. Verify that $y = (a + bx)e^{2x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$.
5. Verify that $y = e^x(A \cos x + B \sin x)$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$. [CBSE 2009]
6. Verify that $y = A \cos 2x - B \sin 2x$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0$. [CBSE 2007, '09]



7. Verify that $y = ae^{2x} + be^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.
8. Show that $y = e^x(A \cos x + B \sin x)$ is the solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.
9. Verify that $y^2 = 4a(x + a)$ is a solution of the differential equation $y \left\{ 1 - \left(\frac{dy}{dx} \right)^2 \right\} = 2x \frac{dy}{dx}$.
10. Verify that $y = ce^{\tan^{-1}x}$ is a solution of the differential equation $(1 + x^2) \frac{d^2y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0$.
11. Verify that $y = ae^{bx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y} \left(\frac{dy}{dx} \right)^2$.
12. Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x} \left(\frac{dy}{dx} \right) = 0$.
13. Verify that $y = e^{-x} + Ax + B$ is a solution of the differential equation $e^x \left(\frac{d^2y}{dx^2} \right) = 1$.
14. Verify that $Ax^2 + By^2 = 1$ is a solution of the differential equation $x \left\{ y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right\} = y \frac{dy}{dx}$.
15. Verify that $y = \frac{c-x}{1+cx}$ is a solution of the differential equation $(1 + x^2) \frac{dy}{dx} + (1 + y^2) = 0$.
16. Verify that $y = \log(x + \sqrt{x^2 + a^2})$ satisfies the differential equation $\frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$.
17. Verify that $y = e^{-3x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.



HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18B)

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Put these values in $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y$ to get 0.

14. Let $Ax^2 + By^2 = 1$ be given. Then,

$$2Ax + 2By \frac{dy}{dx} = 0 \Rightarrow Ax + By \frac{dy}{dx} = 0 \quad \dots (i)$$

$$\Rightarrow Ax = -By \frac{dy}{dx} \Rightarrow \frac{A}{B} = \frac{-y}{x} \cdot \frac{dy}{dx} \quad \dots (ii)$$

On differentiating (i) w.r.t. x , we get

$$A + B \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0 \Rightarrow y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = \frac{-A}{B}$$
$$\Rightarrow \left\{ y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right\} = \frac{y}{x} \cdot \frac{dy}{dx} \quad [\text{from (ii)}].$$

15. Let $c = \tan A$ and $x = \tan B$. Then,

$$y = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B) \Rightarrow \tan^{-1} y = A - B$$
$$\Rightarrow \tan^{-1} y = \tan^{-1}(c) - \tan^{-1} x$$
$$\Rightarrow \frac{1}{(1 + y^2)} \frac{dy}{dx} = 0 - \frac{1}{(1 + x^2)}$$
$$\Rightarrow (1 + x^2) \frac{dy}{dx} + (1 + y^2) = 0.$$

16. $y = \log(x + \sqrt{x^2 + a^2})$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + a^2})} \cdot \left\{ 1 + \frac{2x}{2\sqrt{x^2 + a^2}} \right\} \Rightarrow y_1 = \frac{1}{\sqrt{x^2 + a^2}}$$
$$\Rightarrow y_1^2(x^2 + a^2) = 1 \Rightarrow y_1^2(2x) + (x^2 + a^2)2y_1y_2 = 0$$
$$\Rightarrow (x^2 + a^2)y_2 + xy_1 = 0.$$

Formation of a Differential Equation whose General Solution is Given

METHOD Suppose an equation of a family of curves contains n arbitrary constants (called parameters).

Then, we obtain its differential equation as given below.

Step 1. Differentiate the equation of the given family of curves n times to get n more equations.

Step 2. Eliminate n constants, using these $(n + 1)$ equations.

This gives us the required differential equation of order n .



SOLVED EXAMPLES

EXAMPLE 1 Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant. [CBSE 2013]

SOLUTION The equation of the given family of curves is

$$y = mx \quad \dots (i), \text{ where } m \text{ is a constant.}$$

Since the given equation contains one arbitrary constant, we differentiate it once only.

On differentiating (i) w.r.t. x , we get

$$\frac{dy}{dx} = m. \quad \dots (ii)$$

Putting this value of m from (ii) in (i), we get

$$y = \left(\frac{dy}{dx} \right) x \Rightarrow x \left(\frac{dy}{dx} \right) - y = 0.$$

Hence, $x \left(\frac{dy}{dx} \right) - y = 0$ is the required differential equation.

EXAMPLE 2 Find the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants.

SOLUTION The equation of the given family of curves is

$$y = Ae^x + Be^{-x}. \quad \dots (i)$$

Since the given equation contains two arbitrary constants, we differentiate it two times w.r.t. x .

$$\text{Now, } \frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = y \Rightarrow \frac{d^2y}{dx^2} - y = 0.$$

Hence, $\frac{d^2y}{dx^2} - y = 0$ is the required differential equation.

EXAMPLE 3 Find the differential equation of the family of curves $y = e^x(A \cos x + B \sin x)$, where A and B are arbitrary constants.

SOLUTION The equation of the given family of curves is

$$y = e^x(A \cos x + B \sin x). \quad \dots (i)$$

Since the given equation contains two arbitrary constants, we differentiate it two times w.r.t. x .

$$\text{Now, } \frac{dy}{dx} = e^x(-A \sin x + B \cos x) + e^x(A \cos x + B \sin x)$$

$$\Rightarrow \frac{dy}{dx} - y = e^x(-A \sin x + B \cos x) \quad \dots (ii)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-A \cos x - B \sin x) + e^x(-A \sin x + B \cos x)$$



$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = -y + \left(\frac{dy}{dx} - y \right) \quad [\text{using (i) and (ii)}]$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0, \text{ which is the required differential equation of the given family of curves.}$$

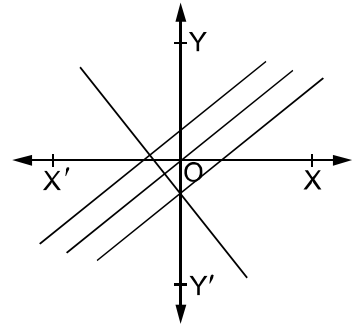
EXAMPLE 4 Find the differential equation of the family of all straight lines.

SOLUTION The general equation of the family of all straight lines is given by $y = mx + c$, where m and c are parameters.

$$\begin{aligned} \text{Now, } y = mx + c &\Rightarrow \frac{dy}{dx} = m \\ &\Rightarrow \frac{d^2y}{dx^2} = 0. \end{aligned}$$

So, the required differential equation is

$$\frac{d^2y}{dx^2} = 0.$$



EXAMPLE 5 Form the differential equation of the family of all circles of radius r .

[CBSE 2010]

SOLUTION The equation of the family of all circles of radius r is

$$(x - a)^2 + (y - b)^2 = r^2, \quad \dots (i)$$

where a and b are arbitrary constants.

Differentiating (i) w.r.t. x , we get

$$2(x - a) + 2(y - b)y_1 = 0$$

$$\Rightarrow (x - a) + (y - b)y_1 = 0. \quad \dots (ii)$$

On differentiating (ii) again w.r.t. x , we get

$$1 + (y - b)y_2 + (y_1)^2 = 0$$

$$\Rightarrow (y - b) = -\frac{\{1 + (y_1)^2\}}{y_2}. \quad \dots (iii)$$

Putting the value of $(y - b)$ from (iii) in (ii), we get

$$(x - a) - \frac{\{1 + (y_1)^2\}y_1}{y_2} = 0 \Rightarrow (x - a) = \frac{\{1 + (y_1)^2\}y_1}{y_2}. \quad \dots (iv)$$

Putting the values of $(y - b)$ and $(x - a)$ from (iii) and (iv) in (i), we get

$$\begin{aligned} &\frac{\{1 + (y_1)^2\}^2 \times (y_1)^2}{(y_2)^2} + \frac{\{1 + (y_1)^2\}^2}{(y_2)^2} = r^2 \\ \Rightarrow &\frac{\{1 + (y_1)^2\}^2 \cdot \{(y_1)^2 + 1\}}{(y_2)^2} = r^2 \\ \Rightarrow &\{1 + (y_1)^2\}^3 = r^2(y_2)^2, \end{aligned}$$

which is the required differential equation.



EXAMPLE 6 Form the differential equation of the family of all circles in first quadrant and touching the coordinate axes. [CBSE 2010]

SOLUTION The general equation of a circle in first quadrant and touching the coordinate axes is given by

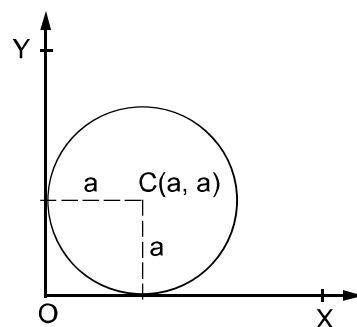
$$(x - a)^2 + (y - a)^2 = a^2 \quad \dots (i), \text{ where } a \text{ is a parameter.}$$

Since this equation contains one parameter, so we will differentiate it only once to get the differential equation. On differentiating (i) w.r.t, x , we get

$$\begin{aligned} 2(x - a) + 2(y - a) \frac{dy}{dx} &= 0 \\ \Rightarrow (x - a) + (y - a) \frac{dy}{dx} &= 0 \\ \Rightarrow (x - a) + (y - a)p &= 0, \text{ where } \frac{dy}{dx} = p \\ \Rightarrow x + yp &= a(1 + p) \Rightarrow a = \frac{(x + yp)}{(1 + p)}. \quad \dots (ii) \end{aligned}$$

Putting the value of a from (ii) in (i), we get

$$\begin{aligned} \left(x - \frac{x + yp}{1 + p}\right)^2 + \left(y - \frac{x + yp}{1 + p}\right)^2 &= \left(\frac{x + yp}{1 + p}\right)^2 \\ \Rightarrow (xp - yp)^2 + (y - x)^2 &= (x + yp)^2 \\ \Rightarrow (x - y)^2 p^2 + (x - y)^2 &= (x + yp)^2 \\ \Rightarrow (x - y)^2 (p^2 + 1) &= (x + yp)^2 \\ \Rightarrow (x - y)^2 \left\{ \left(\frac{dy}{dx}\right)^2 + 1 \right\} &= \left(x + y \frac{dy}{dx}\right)^2, \end{aligned}$$



which is the required differential equation.

EXAMPLE 7 Form the differential equation of the family of all circles touching the x -axis at the origin. [CBSE 2005, '08, '10C]

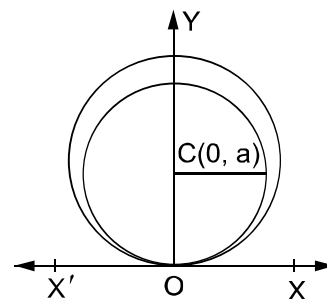
SOLUTION The general equation of a circle touching the x -axis at the origin, is given by

$$\begin{aligned} (x - 0)^2 + (y - a)^2 &= a^2, \text{ where } a \text{ is a parameter} \\ \Rightarrow x^2 + y^2 - 2ay &= 0. \quad \dots (i) \end{aligned}$$

Since this equation contains one parameter, so we shall differentiate it only once to get the differential equation.

On differentiating (i) w.r.t, x , we get

$$\begin{aligned} 2x + 2y \frac{dy}{dx} - 2a \frac{dy}{dx} &= 0 \\ \Rightarrow x + y \frac{dy}{dx} &= a \frac{dy}{dx} \\ \Rightarrow x + yp &= ap \Rightarrow a = \frac{x + yp}{p} \quad \dots (ii), \text{ where } \frac{dy}{dx} = p. \end{aligned}$$



Putting the value of a from (ii) in (i), we get

$$x^2 + y^2 = \frac{2(x + yp) \cdot y}{p}$$

$$\Rightarrow x^2 + y^2 = \frac{2xy}{p} + 2y^2$$

$$\Rightarrow x^2 - y^2 = \frac{2xy}{p}$$

$$\Rightarrow (x^2 - y^2)p = 2xy.$$

Hence, $(x^2 - y^2) \frac{dy}{dx} = 2xy$ is the required differential equation.

EXAMPLE 8 Form the differential equation of the family of all parabolas having vertex at the origin and axis along the positive direction of the x -axis.

SOLUTION The general equation of a parabola having vertex at the origin and axis along the positive direction of the x -axis is given by

$$y^2 = 4ax \quad \dots (i), \text{ where } a \text{ is the parameter.}$$

Since this equation contains one parameter, so we shall differentiate it only once to get the requisite differential equation.

Differentiating (i) w.r.t. x , we get

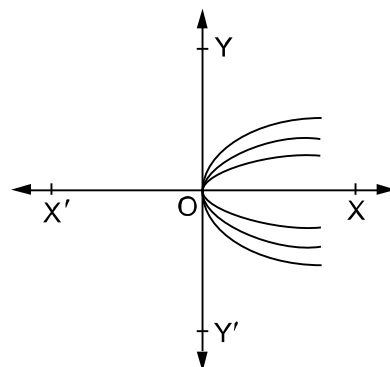
$$2y \frac{dy}{dx} = 4a \Rightarrow y \frac{dy}{dx} = 2a$$

$$\Rightarrow a = \frac{1}{2} y \frac{dy}{dx} \quad \dots (ii)$$

Putting the value of a from (ii) in (i), we get

$$y^2 = 4 \times \frac{1}{2} y \frac{dy}{dx} \times x \Rightarrow y^2 - 2xy \frac{dy}{dx} = 0.$$

Hence, $y^2 - 2xy \frac{dy}{dx} = 0$ is the required differential equation.



EXAMPLE 9 Form the differential equation of the family of all ellipses having foci on the x -axis and centre at the origin. [CBSE 2009C]

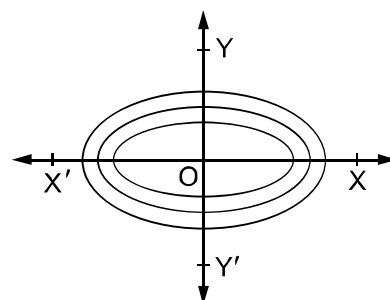
SOLUTION The general equation of an ellipse having foci on the x -axis and centre at the origin, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (i), \text{ where } a \text{ and } b \text{ are the parameters.}$$

Since this equation contains two parameters, so we shall differentiate it twice to get the required differential equation.

Differentiating (i) w.r.t. x , we get:

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{y}{b^2} \cdot \frac{dy}{dx} + \frac{x}{a^2} = 0$$



$$\begin{aligned} \Rightarrow \frac{yy_1}{b^2} &= \frac{-x}{a^2}, \text{ where } \frac{dy}{dx} = y_1 \\ \Rightarrow \frac{yy_1}{x} &= \frac{-b^2}{a^2}. \end{aligned} \quad \dots (ii)$$

Differentiating (ii) w.r.t. x , we get

$$\begin{aligned} \frac{x \cdot \frac{d}{dx}(yy_1) - yy_1 \cdot \frac{d}{dx}(x)}{x^2} &= 0 \\ \Rightarrow x[yy_2 + (y_1)^2] - yy_1 &= 0 \\ \Rightarrow (xy)y_2 + x(y_1)^2 - yy_1 &= 0. \end{aligned}$$

Hence, $(xy)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$ is the required differential equation.

EXAMPLE 10 Form the differential equation for the family of the curves $(y - b)^2 = 4(x - a)$, where a and b are parameters.

SOLUTION The general equation of the given family of curves is

$$(y - b)^2 = 4(x - a) \quad \dots (i), \text{ where } a \text{ and } b \text{ are parameters.}$$

Since the given equation contains two parameters a and b , so we shall differentiate it twice to get the required differential equation.

Differentiating (i) w.r.t. x , we get

$$2(y - b)\frac{dy}{dx} = 4 \Rightarrow (y - b)y_1 = 2 \quad \dots(ii), \text{ where } \frac{dy}{dx} = y_1.$$

Differentiating (ii) w.r.t. x , we get

$$(y - b)y_2 + (y_1)^2 = 0 \quad \dots (iii), \text{ where } \frac{d^2y}{dx^2} = y_2.$$

Putting $(y - b) = \frac{2}{y_1}$ from (ii) in (iii), we get

$$\frac{2y_2}{y_1} + (y_1)^2 = 0 \Rightarrow 2y_2 + (y_1)^3 = 0.$$

Hence, $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is the required differential equation.

EXAMPLE 11 Form the differential equation for the family of the curves $ay^2 = (x - c)^3$, where c is a parameter.

SOLUTION The general equation of the given family of curves is

$$ay^2 = (x - c)^3, \text{ where } c \text{ is a parameter.} \quad \dots (i)$$

Since the given equation has only one parameter, so we shall differentiate it only once to get the required differential equation.

Differentiating (i) w.r.t. x , we get

$$2ayy_1 = 3(x - c)^2. \quad \dots (ii)$$



On dividing (i) and (ii) on each side, we get

$$\frac{ay^2}{2ayy_1} = \frac{(x-c)^3}{3(x-c)^2} \Rightarrow \frac{y}{2y_1} = \frac{(x-c)}{3} \Rightarrow (x-c) = \frac{3y}{2y_1}. \quad \dots \text{(iii)}$$

Putting the value of $(x-c)$ from (iii) in (i), we get

$$ay^2 = \left(\frac{3y}{2y_1} \right)^3 \Rightarrow 8ay_1^3 = 27y.$$

Hence, $8a \left(\frac{dy}{dx} \right)^3 - 27y = 0$ is the required differential equation.

EXAMPLE 12 Form the differential equation for the family of the curves $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants.

SOLUTION The general equation of the given family of curves is

$$y^2 = a(b^2 - x^2) \quad \dots \text{(i)}, \text{ where } a \text{ and } b \text{ are the parameters.}$$

Since the given equation has two parameters, so we shall have to differentiate it two times to get the required differential equations.

Differentiating (i) w.r.t. x we get

$$2yy_1 = -2ax \Rightarrow yy_1 = -ax. \quad \dots \text{(ii)}$$

On differentiating (ii) w.r.t. x , we get

$$yy_2 - (y_1)^2 = -a$$

$$\Rightarrow yy_2 - (y_1)^2 = \frac{yy_1}{x} \quad [\text{using (ii)}]$$

$$\Rightarrow (xy)y_2 - x(y_1)^2 - yy_1 = 0.$$

Hence, $(xy) \frac{d^2y}{dx^2} - x \left(\frac{dy}{dx} \right)^2 - y \left(\frac{dy}{dx} \right) = 0$ is the required differential equation.

EXERCISE 18C

- Form the differential equation of the family of straight lines $y = mx + c$, where m and c are arbitrary constants.
- Form the differential equation of the family of concentric circles $x^2 + y^2 = a^2$, where $a > 0$ and a is a parameter.
- Form the differential equation of the family of curves, $y = a \sin (bx + c)$, where a and c are parameters.
- Form the differential equation of the family of curves $x = A \cos nt + B \sin nt$, where A and B are arbitrary constants. [CBSE 2007]
- Form the differential equation of the family of curves $y = ae^{bx}$, where a and b are arbitrary constants.
- Form the differential equation of the family of curves $y^2 = m(a^2 - x^2)$, where a and m are parameters.



7. Form the differential equation of the family of curves given by $(x - a)^2 + 2y^2 = a^2$, where a is an arbitrary constant.
8. Form the differential equation of the family of curves given by $x^2 + y^2 - 2ay = a^2$, where a is an arbitrary constant. [CBSE 2005]
9. Form the differential equation of the family of all circles touching the y -axis at the origin. [CBSE 2008C, '10C]
10. Form the differential equation of the family of circles having centres on the y -axis and radius 2 units.
11. Form the differential equation of the family of circles in second quadrant and touching the coordinate axes.
12. Form the differential equation of the family of circles having centres on the x -axis and radius unity.
13. Form the differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$, where a is the parameter.
14. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y -axis. [CBSE 2010, '11]
15. Form the differential equation of the family of ellipses having foci on the y -axis and centre at the origin.
16. Form the differential equation of the family of hyperbolas having foci on the x -axis and centre at the origin.

ANSWERS (EXERCISE 18C)

1. $\frac{d^2y}{dx^2} = 0$
2. $x + y \frac{dy}{dx} = 0$
3. $\frac{d^2y}{dx^2} + b^2y = 0$
4. $\frac{d^2x}{dt^2} + n^2x = 0$
5. $\left(\frac{dy}{dx}\right)^2 = y \cdot \frac{d^2y}{dx^2}$
6. $xy \frac{d^2y}{dx^2} - x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} = 0$
7. $\frac{dy}{dx} = \frac{2y^2 - x^2}{4xy}$
8. $p^2(x^2 - 2y^2) - 4pxy - x^2 = 0$, where $p = \frac{dy}{dx}$
9. $y^2 - x^2 - 2xy \frac{dy}{dx} = 0$
10. $x^2 \left\{ 1 + \frac{1}{(dy/dx)^2} \right\} = 4$
11. $(x + y)^2 \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} = \left(x + y \frac{dy}{dx}\right)^2$
12. $y^2 \left(\frac{dy}{dx}\right)^2 + y^2 = 1$
13. $(x^2 - y^2 - a^2) \frac{dy}{dx} = 2xy$
14. $x \frac{dy}{dx} = 2y$
15. $xy \left(\frac{d^2y}{dx^2}\right) + x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} = 0$
16. $xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} = 0$



HINTS TO SOME SELECTED QUESTIONS (EXERCISE 18C)

6. See Example 12.

7. Given family of curves is $x^2 + 2y^2 - 2ax = 0$ (i)

Differentiating (i) w.r.t. x , we get

$$2x + 4y \frac{dy}{dx} - 2a = 0 \Rightarrow a = \left(2y \frac{dy}{dx} + x \right).$$

Put this value of a in (i).

8. Given family of curves is $x^2 + y^2 - 2ay = a^2$ (i)

Differentiating (i) w.r.t. x , we get

$$2x + 2y \frac{dy}{dx} - 2a \frac{dy}{dx} = 0 \Rightarrow x + y \frac{dy}{dx} = a \frac{dy}{dx}$$

$$\Rightarrow a = \frac{x + yp}{p}, \text{ where } p = \frac{dy}{dx}.$$

Putting $a = \frac{x + yp}{p}$ in (i), we get:

$$x^2 + y^2 - \frac{2(x + yp)y}{p} = \left(\frac{x + yp}{p} \right)^2$$

$$\Rightarrow (x^2 + y^2)p^2 - 2yp(x + yp) = (x + yp)^2$$

$$\Rightarrow x^2p^2 + y^2p^2 - 2xyp - 2y^2p^2 = x^2 + y^2p^2 + 2xyp$$

$$\Rightarrow x^2p^2 - 2y^2p^2 - 4xyp - x^2 = 0$$

$$\Rightarrow (x^2 - 2y^2)p^2 - 4xyp - x^2 = 0, \text{ where } p = \frac{dy}{dx}.$$

9. The equation of given family of circles is

$$(x - a)^2 + (y - 0)^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0. \dots (i)$$

Differentiating (i) w.r.t. x , we get

$$2x + 2y \frac{dy}{dx} - 2a = 0 \Rightarrow x + y \frac{dy}{dx} = a. \dots (ii)$$

Put the value of a from (ii) in (i), we get the desired differential equation.

10. The given family is $x^2 + (y - a)^2 = 4$.

11. The equation of the given family of circles is

$$(x + a)^2 + (y - a)^2 = a^2 \dots (i),$$

where a is a parameter.

Differentiating (i) w.r.t. x , we get

$$2(x + a) + 2(y - a) \frac{dy}{dx} = 0 \Rightarrow (x + a) + (y - a)p = 0$$

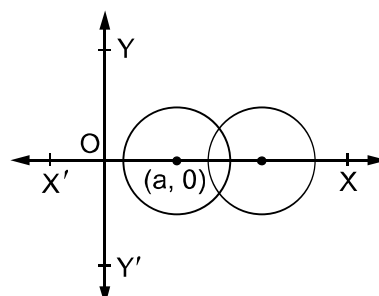
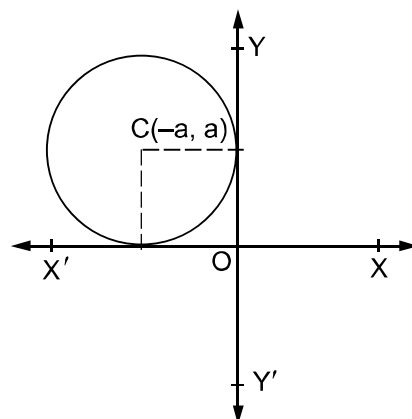
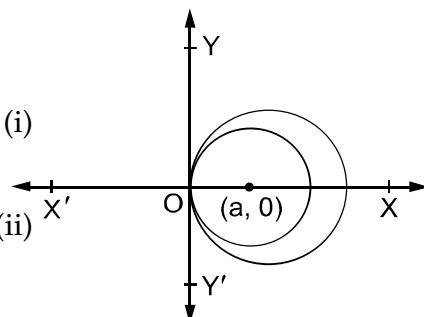
$$\Rightarrow a = \frac{(x + yp)}{(p - 1)}.$$

Put this value of a in (i) to get the required equation.

12. The equation of the given family of circles with centre $(a, 0)$ and radius 1, is given by

$$(x - a)^2 + (y - 0)^2 = 1^2, \text{ where } a \text{ is the parameter}$$

$$\Rightarrow (x - a)^2 + y^2 = 1. \dots (i)$$



13. The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \dots (i)$$

If it passes through the points $A(-a, 0)$ and $B(a, 0)$, we have

$$a^2 - 2ga + c = 0 \quad \dots (ii) \text{ and } a^2 + 2ga + c = 0 \quad \dots (iii)$$

Adding (ii) and (iii), we get $2(a^2 + c) = 0 \Rightarrow a^2 + c = 0 \Rightarrow c = -a^2$.

Putting $c = -a^2$ in (iii), we get $2ga = 0 \Rightarrow g = 0$.

Putting $g = 0$ and $c = -a^2$ in (i), we get

$$x^2 + y^2 + 2fy - a^2 = 0 \quad \dots (iv), \text{ where } f \text{ is the parameter.}$$

Differentiating (iv) w.r.t. x , we get

$$2x + 2yy_1 + 2fy_1 = 0 \Rightarrow fy_1 = -(x + yy_1) \Rightarrow f = \frac{-(x + yy_1)}{y_1}.$$

Putting this value of f in (iv), we get

$$x^2 + y^2 - \frac{2y(x + yy_1)}{y_1} - a^2 = 0$$

$$\Rightarrow x^2 y_1 + y^2 y_1 - 2xy - 2y^2 y_1 - a^2 y_1 = 0 \Rightarrow (x^2 - y^2 - a^2) y_1 = 2xy.$$

14. The equation of the given family of parabolas is given by $x^2 = 4ay$.

15. The equation of the family of ellipses having centre at the origin and foci on the y -axis, is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (i), \text{ where } b > a \text{ and } a, b \text{ are the parameters.}$$

Differentiating (i), w.r.t. x , we get

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{x}{a^2} + \frac{yy_1}{b^2} = 0. \quad \dots (ii)$$

Differentiating (ii) w.r.t. x , we get

$$\frac{1}{a^2} + \frac{yy_2}{b^2} + \frac{y_1^2}{b^2} = 0 \Rightarrow \frac{x}{a^2} + \frac{xyy_2}{b^2} + \frac{xy_1^2}{b^2} = 0. \quad \dots (iii)$$

Subtracting (ii) from (iii), we get

$$\frac{1}{b^2} \{xyy_2 + xy_1^2 - yy_1\} = 0 \Rightarrow xy \left(\frac{d^2 y}{dx^2} \right) + x \left(\frac{dy}{dx} \right)^2 - y \left(\frac{dy}{dx} \right) = 0.$$

16. The equation of the family of hyperbolas having foci on the x -axis and centre at the origin is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (i), \text{ where } a \text{ and } b \text{ are the parameters.}$$

Differentiating (i), w.r.t. x , we get

$$\frac{2x}{a^2} - \frac{2yy_1}{b^2} = 0 \Rightarrow \frac{x}{a^2} - \frac{yy_1}{b^2} = 0. \quad \dots (ii)$$

Differentiating (ii), w.r.t. x , we get

$$\frac{1}{a^2} - \frac{yy_2}{b^2} - \frac{y_1^2}{b^2} = 0. \quad \dots (iii)$$

Multiplying (iii) by x and subtracting from (ii), we get

$$\frac{1}{b^2} \{xyy_2 + xy_1^2 - yy_1\} = 0 \Rightarrow xy \left(\frac{d^2 y}{dx^2} \right) + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0.$$

19. DIFFERENTIAL EQUATIONS WITH VARIABLE SEPARABLE

Method of Solving Differential Equations with Variables Separable

GENERAL SOLUTION Let the given differential equation be expressed in the form $g(y)dy = f(x)dx$.

Then, the **general solution** of the above differential equation is given by $\int g(y)dy = \int f(x)dx + C$, where C is an arbitrary constant.

PARTICULAR SOLUTION Let $x = a$ be given and let the corresponding value of y be given as b . Putting $x = a$ and $y = b$ in the general solution, we get the value of C . With this value of C , we get the particular solution of the given differential equation.

SOLVED EXAMPLES

EXAMPLE 1 Find the general solution of the differential equation

$$(x + 2) \frac{dy}{dx} = x^2 + 5x - 3 \quad (x \neq -2).$$

SOLUTION The given differential equation may be written as

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2 + 5x - 3}{x + 2} \\ \Rightarrow dy &= \left(\frac{x^2 + 5x - 3}{x + 2} \right) dx \quad [\text{separating the variables}] \\ \Rightarrow \int dy &= \int \left(\frac{x^2 + 5x - 3}{x + 2} \right) dx \\ \Rightarrow y &= \int \left\{ x + 3 - \frac{9}{(x + 2)} \right\} dx + C, \text{ where } C \text{ is an arbitrary constant} \\ &\quad [\text{on dividing } (x^2 + 5x - 3) \text{ by } (x + 2)] \\ \Rightarrow y &= \frac{x^2}{2} + 3x - 9 \log |x + 2| + C. \end{aligned}$$

Hence, $y = \frac{x^2}{2} + 3x - 9 \log |x + 2| + C$ is the required general solution.

EXAMPLE 2 Find the general solution of the differential equation

$$(1 + x^2) \frac{dy}{dx} - x = 2 \tan^{-1} x. \quad [\text{CBSE 2007}]$$



SOLUTION The given differential equation may be written as

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\tan^{-1}x}{(1+x^2)} + \frac{x}{(1+x^2)} \\ \Rightarrow dy &= \left\{ \frac{2\tan^{-1}x}{(1+x^2)} + \frac{x}{(1+x^2)} \right\} dx \quad [\text{separating the variables}] \\ \Rightarrow \int dy &= \int \frac{2\tan^{-1}x}{(1+x^2)} dx + \int \frac{x}{(1+x^2)} dx + C, \\ &\text{where } C \text{ is an arbitrary constant} \\ \Rightarrow y &= 2 \int t dt + \frac{1}{2} \int \frac{2x}{(1+x^2)} dx + C \\ &\quad [\text{putting } \tan^{-1}x = t \text{ and } \frac{1}{(1+x^2)} dx = dt \text{ in 1st integral}] \\ \Rightarrow y &= t^2 + \frac{1}{2} \log |1+x^2| + C \\ \Rightarrow y &= (\tan^{-1}x)^2 + \frac{1}{2} \log |1+x^2| + C. \\ \text{Hence, } y &= (\tan^{-1}x)^2 + \frac{1}{2} \log |1+x^2| + C \text{ is the required solution.}\end{aligned}$$

EXAMPLE 3 Find the general solution of the differential equation $\frac{dy}{dx} = \log(x+1)$.

[CBSE 2006]

SOLUTION We have

$$\begin{aligned}\frac{dy}{dx} &= \log(x+1) \\ \Rightarrow dy &= \log(x+1) dx \\ \Rightarrow \int dy &= \int \log(x+1) dx \quad [\text{integrating both sides}] \\ \Rightarrow y &= \int \underbrace{\log(x+1)}_I \cdot \underbrace{1}_{II} dx + C, \text{ where } C \text{ is an arbitrary constant} \\ &= \{\log(x+1) \cdot x\} - \int \frac{1}{(x+1)} \cdot x dx + C \quad [\text{integrating by parts}] \\ &= x \log(x+1) - \int \frac{(x+1)-1}{(x+1)} dx + C \\ &= x \log(x+1) - \int \left\{ 1 - \frac{1}{(x+1)} \right\} dx + C \\ &= x \log(x+1) - x + \log(x+1) + C \\ &= (x+1) \log(x+1) - x + C.\end{aligned}$$

Hence, $y = (x+1) \log(x+1) - x + C$ is the required solution.

EXAMPLE 4 Find the general solution of the differential equation

$$\frac{dy}{dx} = \sin^{-1}x.$$



SOLUTION We have

$$\begin{aligned}\frac{dy}{dx} &= \sin^{-1}x \\ \Rightarrow dy &= \sin^{-1}x \, dx \quad [\text{separating the variables}] \\ \Rightarrow \int dy &= \int \sin^{-1}x \, dx \quad [\text{integrating both sides}] \\ \Rightarrow y &= \int \underbrace{(\sin^{-1}x)}_I \cdot \underbrace{1}_{II} \, dx + C, \text{ where } C \text{ is an arbitrary constant} \\ &= (\sin^{-1}x) \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x \, dx + C \quad [\text{integrating by parts}] \\ &= (\sin^{-1}x) \cdot x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx + C \\ &= (\sin^{-1}x) \cdot x + \frac{1}{2} \int \frac{1}{\sqrt{t}} \, dt + C, \text{ where } (1-x^2) = t \\ &= (\sin^{-1}x) \cdot x + \frac{1}{2} \times 2\sqrt{t} + C \\ &= (\sin^{-1}x)x + \sqrt{1-x^2} + C.\end{aligned}$$

Hence, $y = (\sin^{-1}x)x + \sqrt{1-x^2} + C$ is the required solution.

Hence, $y = x(\sin^{-1}x) + \sqrt{1-x^2} + C$ is the required solution of the given differential equation.

EXAMPLE 5 Find the general solution of the differential equation

$$\frac{dy}{dx} = \sqrt{4-y^2} \quad (-2 < y < 2). \quad [\text{CBSE 2002C}]$$

SOLUTION We have $\frac{dy}{dx} = \sqrt{4-y^2}$

$$\begin{aligned}\Rightarrow \frac{dy}{\sqrt{4-y^2}} &= dx \quad [\text{on separating the variables}] \\ \Rightarrow \int \frac{dy}{\sqrt{2^2-y^2}} &= \int dx \quad [\text{integrating both sides}] \\ \Rightarrow \sin^{-1}\left(\frac{y}{2}\right) &= x + C, \text{ where } C \text{ is an arbitrary constant.}\end{aligned}$$

Hence, $\sin^{-1}\left(\frac{y}{2}\right) = x + C$ is the required solution.

EXAMPLE 6 Find the general solution of the differential equation

$$(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x. \quad [\text{CBSE 2010}]$$

SOLUTION The given differential equation may be written as

$$\frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} = \frac{(2x^2 + x)}{x^2(x+1) + (x+1)} = \frac{(2x^2 + x)}{(x+1)(x^2 + 1)}$$



$$\Rightarrow dy = \left\{ \frac{(2x^2 + x)}{(x+1)(x^2+1)} \right\} dx$$

$$\Rightarrow \int dy = \int \frac{(2x^2 + x)}{(x+1)(x^2+1)} dx \quad \dots (i) \quad [\text{integrating both sides}]$$

$$\Rightarrow y = \int \frac{(2x^2 + x)}{(x+1)(x^2+1)} dx + C, \text{ where } C \text{ is an arbitrary constant.}$$

$$\text{Let } \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}.$$

$$\text{Then, } 2x^2 + x \equiv A(x^2 + 1) + (Bx + C)(x + 1). \quad \dots (ii)$$

$$\text{Putting } x = -1 \text{ in (ii), we get } 2A = 1 \Rightarrow A = \frac{1}{2}.$$

$$\text{Putting } x = 0 \text{ in (ii), we get } A + C = 0 \Rightarrow C = -A = \frac{-1}{2}.$$

$$\text{Putting } x = 1 \text{ in (ii), we get } 2A + 2B + 2C = 3.$$

$$\therefore A + B + C = \frac{3}{2} \Rightarrow \frac{1}{2} + B - \frac{1}{2} = \frac{3}{2} \Rightarrow B = \frac{3}{2}.$$

$$\therefore A = \frac{1}{2}, B = \frac{3}{2} \text{ and } C = \frac{-1}{2}.$$

$$\therefore y = \frac{1}{2} \int \frac{dx}{(x+1)} + \int \frac{\left(\frac{3}{2}x - \frac{1}{2}\right)}{(x^2+1)} dx + C$$

$$\Rightarrow y = \frac{1}{2} \int \frac{dx}{(x+1)} + \frac{3}{2} \int \frac{x}{(x^2+1)} dx - \frac{1}{2} \int \frac{dx}{(x^2+1)} + C$$

$$\Rightarrow y = \frac{1}{2} \log |x+1| + \frac{3}{4} \int \frac{2x}{(x^2+1)} dx - \frac{1}{2} \tan^{-1} x + C$$

$$\Rightarrow y = \frac{1}{2} \log |x+1| + \frac{3}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C.$$

Hence, $y = \frac{1}{2} \log |x+1| + \frac{3}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C$ is the required solution.

EXAMPLE 7 Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}.$$

SOLUTION $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$$\Rightarrow \frac{1}{(1+y^2)} dy = \frac{1}{(1+x^2)} dx \quad [\text{separating the variables}]$$



$$\Rightarrow \int \frac{1}{(1+y^2)} dy = \int \frac{1}{(1+x^2)} dx$$

$$\Rightarrow \tan^{-1}y = \tan^{-1}x + C_1, \text{ where } C_1 \text{ is an arbitrary constant}$$

$$\Rightarrow \tan^{-1}y - \tan^{-1}x = C_1$$

$$\Rightarrow \tan^{-1}\left(\frac{y-x}{1+yx}\right) = C_1$$

$$\Rightarrow \frac{y-x}{1+yx} = \tan C_1 = C, \text{ where } C = \tan C_1.$$

$$\text{Hence, } \frac{y-x}{1+yx} = C \text{ is the required solution.}$$

EXAMPLE 8 Find the general solution of the differential equation

$$\log \left(\frac{dy}{dx} \right) = (ax + by).$$

SOLUTION $\log \left(\frac{dy}{dx} \right) = ax + by$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by} = e^{ax} \cdot e^{by}$$

$$\Rightarrow \frac{1}{e^{by}} dy = e^{ax} dx \quad [\text{on separating the variables}]$$

$$\Rightarrow \int e^{-by} dy = \int e^{ax} dx \quad [\text{integrating both sides}]$$

$$\Rightarrow \frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C$$

$$\Rightarrow ae^{-by} + be^{ax} = C', \text{ where } C' = -abC.$$

$$\text{Thus, } ae^{-by} + be^{ax} = C' \text{ is the required solution.}$$

EXAMPLE 9 Find the general solution of the differential equation

$$\sqrt{1+x^2+y^2+x^2y^2} + xy \frac{dy}{dx} = 0. \quad [\text{CBSE 2010}]$$

SOLUTION The given differential equation may be written as

$$\sqrt{(1+x^2)+y^2(1+x^2)} + xy \frac{dy}{dx} = 0$$

$$\Rightarrow \sqrt{(1+x^2)(1+y^2)} + xy \frac{dy}{dx} = 0$$

$$\Rightarrow (\sqrt{1+x^2})(\sqrt{1+y^2}) + xy \frac{dy}{dx} = 0$$

$$\Rightarrow xy \frac{dy}{dx} = -(\sqrt{1+x^2})(\sqrt{1+y^2})$$

$$\Rightarrow \frac{y}{\sqrt{1+y^2}} dy = \frac{-\sqrt{1+x^2}}{x} dx$$



$$\Rightarrow \int \frac{y}{\sqrt{1+y^2}} dy = -\int \frac{\sqrt{1+x^2}}{x^2} \cdot x dx. \quad \dots (i)$$

Putting $1+x^2=u^2$ and $1+y^2=v^2$, we get

$$2x dx = 2u du \text{ and } 2y dy = 2v dv$$

$$\Rightarrow x dx = u du \text{ and } y dy = v dv.$$

Substituting these values in (i), we get

$$\int \frac{v dv}{v} = -\int \frac{u \times u du}{(u^2-1)}$$

$$\Rightarrow \int dv = -\int \frac{u^2}{(u^2-1)} du = -\int \frac{(u^2-1)+1}{(u^2-1)} du$$

$$\Rightarrow v = -\int \left\{ 1 + \frac{1}{(u^2-1)} \right\} du + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow v = -u - \frac{1}{2} \log \left| \frac{u-1}{u+1} \right| + C$$

$$\Rightarrow \sqrt{1+y^2} = -\sqrt{1+x^2} - \frac{1}{2} \log \left| \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} \right| + C,$$

which is the required solution.

EXAMPLE 10 Find the general solution of the differential equation

$$(x \cos y) dy = e^x (x \log x + 1) dx.$$

[CBSE 2007]

SOLUTION The given differential equation may be written as

$$\cos y dy = \left(e^x \log x + \frac{e^x}{x} \right) dx$$

$$\Rightarrow \int \cos y dy = \int (\log x) e^x dx + \int \frac{e^x}{x} dx + C,$$

where C is an arbitrary constant

$$\Rightarrow \sin y = (\log x) e^x - \int \frac{1}{x} \cdot e^x dx + \int \frac{e^x}{x} dx + C \text{ [integrating by parts]}$$

$$\Rightarrow \sin y = (\log x) e^x + C, \text{ which is the required solution.}$$

EXAMPLE 11 Find the general solution of the differential equation

$$x\sqrt{1-y^2} dx + y\sqrt{1-x^2} dy = 0.$$

SOLUTION We have

$$x\sqrt{1-y^2} dx + y\sqrt{1-x^2} dy = 0$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} dx + \frac{y}{\sqrt{1-y^2}} dy = 0 \text{ [on separating the variables]}$$



$$\begin{aligned}
&\Rightarrow \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{y}{\sqrt{1-y^2}} dy = C \quad [\text{integrating both sides}] \\
&\Rightarrow -\frac{1}{2} \cdot \int \frac{(-2x)}{\sqrt{1-x^2}} dx - \frac{1}{2} \cdot \int \frac{(-2y)}{\sqrt{1-y^2}} dy = C \\
&\Rightarrow -\frac{1}{2} \cdot \int \frac{1}{\sqrt{t}} dt - \frac{1}{2} \cdot \int \frac{1}{\sqrt{s}} ds = C, \text{ where } (1-x^2) = t \text{ and } (1-y^2) = s \\
&\Rightarrow -\frac{1}{2} \int t^{-1/2} dt - \frac{1}{2} \int s^{-1/2} ds = C \\
&\Rightarrow -\sqrt{t} - \sqrt{s} = C \\
&\Rightarrow \sqrt{t} + \sqrt{s} = k \quad [\text{where } k = -C] \\
&\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = k. \\
&\text{Hence, } \sqrt{1-x^2} + \sqrt{1-y^2} = k \text{ is the required solution.}
\end{aligned}$$

EXAMPLE 12 Find the general solution of the differential equation

$$y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right). \quad [\text{CBSE 2008}]$$

SOLUTION We have

$$\begin{aligned}
&y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right) \\
&\Rightarrow (y - ay^2) = (a + x) \frac{dy}{dx} \\
&\Rightarrow \frac{dy}{y(1-ay)} = \frac{dx}{(a+x)} \quad [\text{on separating the variables}] \\
&\Rightarrow \int \frac{dy}{y(1-ay)} = \int \frac{dx}{(a+x)} \quad [\text{integrating both sides}] \\
&\Rightarrow \int \left(\frac{1}{y} + \frac{a}{1-ay} \right) dy = \int \frac{dx}{(a+x)} \quad [\text{by partial fractions}] \\
&\Rightarrow \log |y| - \log |1-ay| = \log |a+x| + \log |C_1|, \\
&\hspace{15em} \text{where } C_1 \text{ is an arbitrary constant} \\
&\Rightarrow \log \left| \frac{y}{(1-ay)(a+x)} \right| = \log |C_1| \\
&\Rightarrow \frac{y}{(1-ay)(a+x)} = \pm C_1 = C \text{ (say).}
\end{aligned}$$

Hence, $y = C(1-ay)(a+x)$ is the required solution.

EXAMPLE 13 Find the general solution of the differential equation

$$(\sqrt{a+x}) \frac{dy}{dx} + x = 0.$$

SOLUTION We have

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a+x}}$$



$$\begin{aligned}
 \Rightarrow dy &= \frac{-x}{\sqrt{a+x}} dx \quad [\text{on separating the variables}] \\
 \Rightarrow \int dy &= \int \frac{-x}{\sqrt{a+x}} dx \quad [\text{integrating both sides}] \\
 \Rightarrow y &= -\int \frac{[(a+x)-a]}{\sqrt{a+x}} dx \\
 \Rightarrow y &= -\int \left(\sqrt{a+x} - \frac{a}{\sqrt{a+x}} \right) dx \\
 \Rightarrow y &= -\int \sqrt{a+x} dx + a \int (a+x)^{-1/2} dx \\
 \Rightarrow y &= -\frac{2}{3}(a+x)^{3/2} + 2a\sqrt{a+x} + C, \text{ which is the required solution.}
 \end{aligned}$$

EXAMPLE 14 Solve the differential equation

$$x \cos y dy = (xe^x \log x + e^x) dx.$$

SOLUTION We have

$$\begin{aligned}
 x \cos y dy &= (xe^x \log x + e^x) dx \\
 \Rightarrow \cos y dy &= e^x \left(\log x + \frac{1}{x} \right) dx \quad [\text{on separating the variables}] \\
 \Rightarrow \int \cos y dy &= \int e^x \left(\log x + \frac{1}{x} \right) dx \quad [\text{integrating both sides}] \\
 \Rightarrow \sin y &= e^x \log x + C \quad [\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x)].
 \end{aligned}$$

Hence, $\sin y = e^x (\log x) + C$ is the required solution.

EXAMPLE 15 Solve the differential equation

$$\frac{dy}{dx} = \frac{e^x (\sin^2 x + \sin 2x)}{y(2 \log y + 1)}.$$

SOLUTION We have

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{e^x (\sin^2 x + \sin 2x)}{y(2 \log y + 1)} \\
 \Rightarrow y(2 \log y + 1) dy &= e^x (\sin^2 x + \sin 2x) dx \\
 \Rightarrow \int y \log y dy + \int y dy &= \int e^x (\sin^2 x + \sin 2x) dx \\
 \Rightarrow 2 \left[(\log y) \cdot \frac{y^2}{2} - \int \frac{1}{y} \cdot \frac{y^2}{2} dy \right] + \frac{1}{2} y^2 &= \int e^x (\sin^2 x + \sin 2x) dx \\
 & \quad [\text{integrating by parts}] \\
 \Rightarrow y^2 (\log y) - \int y dy + \frac{1}{2} y^2 &= \int e^x (\sin^2 x + \sin 2x) dx \\
 \Rightarrow y^2 (\log y) &= e^x \sin^2 x + C \quad [\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C]. \\
 \therefore y^2 (\log y) &= e^x \sin^2 x + C \text{ is the required solution.}
 \end{aligned}$$



EXAMPLE 16 Solve the differential equation

$$(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0.$$

SOLUTION $(1+x)(1+y^2)dx + (1+y)(1+x^2)dy = 0$

$$\Rightarrow \frac{(1+x)}{(1+x^2)}dx + \frac{(1+y)}{(1+y^2)}dy = 0 \quad [\text{on separating the variables}]$$

$$\Rightarrow \int \frac{(1+x)}{(1+x^2)}dx + \int \frac{(1+y)}{(1+y^2)}dy = C \quad [\text{integrating both sides}]$$

$$\Rightarrow \int \left\{ \frac{1}{(1+x^2)} + \frac{x}{(1+x^2)} \right\} dx + \int \left\{ \frac{1}{(1+y^2)} + \frac{y}{(1+y^2)} \right\} dy = C$$

$$\Rightarrow \int \frac{1}{(1+x^2)}dx + \frac{1}{2} \cdot \int \frac{2x}{(1+x^2)}dx + \int \frac{1}{(1+y^2)}dy + \frac{1}{2} \cdot \int \frac{2y}{(1+y^2)}dy = C$$

$$\Rightarrow \tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = C$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}y + \frac{1}{2}\{\log(1+x^2) + \log(1+y^2)\} = C,$$

which is the required solution.

EXAMPLE 17 Solve the differential equation

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2 y^2 = 0.$$

[CBSE 2014]

SOLUTION The given differential equation may be written as

$$\operatorname{cosec} x \log y \frac{dy}{dx} = -x^2 y^2$$

$$\Rightarrow \frac{\log y}{y^2} dy = -x^2 \sin x dx \quad [\text{separating the variables}]$$

$$\Rightarrow \int (\log y) \left(\frac{1}{y^2} \right) dy = - \int x^2 \sin x dx \quad [\text{integrating both sides}].$$

Integrating by parts on each side, we get:

$$(\log y) \left(\frac{-1}{y} \right) - \int \frac{1}{y} \cdot \left(\frac{-1}{y} \right) dy = -[x^2(-\cos x) - \int 2x(-\cos x) dx] + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow \frac{-\log y}{y} + \int \frac{1}{y^2} dy = x^2 \cos x - 2 \int x \cos x dx + C_1$$

$$\Rightarrow \frac{-\log y}{y} - \frac{1}{y} = x^2 \cos x - 2x \sin x + 2 \int \sin x dx + C_1$$

[integrating by parts]

$$\Rightarrow \frac{-\log y}{y} - \frac{1}{y} = x^2 \cos x - 2x \sin x - 2 \cos x + C_1$$

$$\Rightarrow \frac{\log y}{y} + \frac{1}{y} = -x^2 \cos x + 2x \sin x + 2 \cos x + C, \text{ where } -C_1 = C.$$

This is the required solution of the given differential equation.



EXAMPLE 18 Show that the general solution of the differential equation $\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$ is given by $(x + y + 1) = C(1 - x - y - 2xy)$, where C is an arbitrary constant.

SOLUTION We have

$$\begin{aligned} \frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{(y^2 + y + 1)}{(x^2 + x + 1)} \\ \Rightarrow \frac{dy}{(y^2 + y + 1)} &= -\frac{dx}{(x^2 + x + 1)} \\ \Rightarrow \int \frac{dy}{(y^2 + y + 1)} &= -\int \frac{dx}{(x^2 + x + 1)} \quad [\text{on integrating both sides}] \\ \Rightarrow \int \frac{dy}{\left\{\left(y + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}} &= -\int \frac{dx}{\left\{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}} \\ \Rightarrow \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \left(\frac{y + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) &= \frac{-1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C_1, \end{aligned}$$

where C_1 is an arbitrary constant

$$\begin{aligned} \Rightarrow \left(\frac{2}{\sqrt{3}} \right) \tan^{-1} \left(\frac{2y + 1}{\sqrt{3}} \right) + \left(\frac{2}{\sqrt{3}} \right) \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right) &= C_1 \\ \Rightarrow \tan^{-1} \left(\frac{2y + 1}{\sqrt{3}} \right) + \tan^{-1} \left(\frac{2x + 1}{\sqrt{3}} \right) &= \frac{\sqrt{3}}{2} C_1 \\ \Rightarrow \tan^{-1} a + \tan^{-1} b = \frac{\sqrt{3}}{2} C_1, \text{ where } \left(\frac{2y + 1}{\sqrt{3}} \right) = a \text{ and } \left(\frac{2x + 1}{\sqrt{3}} \right) = b \\ \Rightarrow \tan^{-1} \left(\frac{a + b}{1 - ab} \right) &= \frac{\sqrt{3}}{2} C_1, \end{aligned}$$

$$\text{where } (a + b) = \frac{2(x + y + 1)}{\sqrt{3}} \text{ and } ab = \frac{(4xy + 2x + 2y + 1)}{3}$$

$$\begin{aligned} \Rightarrow \left(\frac{a + b}{1 - ab} \right) &= \tan \left(\frac{\sqrt{3}}{2} C_1 \right) \\ \Rightarrow \frac{\left\{ \frac{2(x + y + 1)}{\sqrt{3}} \right\}}{\left\{ 1 - \frac{(4xy + 2x + 2y + 1)}{3} \right\}} &= \tan \left(\frac{\sqrt{3}}{2} C_1 \right) \end{aligned}$$



$$\Rightarrow \frac{(x+y+1)}{(1-x-y-2xy)} = \frac{1}{\sqrt{3}} \tan \left(\frac{\sqrt{3}}{2} C_1 \right) = C \text{ (say)}$$

$$\Rightarrow (x+y+1) = C(1-x-y-2xy),$$

which is the solution of the given DE.

EXAMPLE 19 Solve the differential equation

$$x(1+y^2)dx - y(1+x^2)dy = 0,$$

given that $y = 0$ when $x = 1$.

[CBSE 2006C, '14]

SOLUTION The given differential equation is

$$x(1+y^2)dx - y(1+x^2)dy = 0$$

$$\Rightarrow x(1+y^2)dx = y(1+x^2)dy$$

$$\Rightarrow \frac{y}{(1+y^2)} dy = \frac{x}{(1+x^2)} dx$$

$$\Rightarrow \int \frac{y}{(1+y^2)} dy = \int \frac{x}{(1+x^2)} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2y}{(1+y^2)} dy = \frac{1}{2} \int \frac{2x}{(1+x^2)} dx$$

$$\Rightarrow \frac{1}{2} \log (1+y^2) = \frac{1}{2} \log (1+x^2) + \frac{1}{2} \log |C_1|,$$

where C_1 is an arbitrary constant

$$\Rightarrow \log (1+y^2) = \log (1+x^2) + \log |C_1|$$

$$\Rightarrow \frac{(1+y^2)}{(1+x^2)} = \pm C_1 = C \text{ (say)}$$

$$\Rightarrow (1+y^2) = C(1+x^2). \quad \dots (i)$$

Putting $x = 1$ and $y = 0$ in (i), we get $C = \frac{1}{2}$.

$$\therefore (1+y^2) = \frac{1}{2}(1+x^2) \Rightarrow (x^2 - 2y^2) = 1.$$

Hence, $(x^2 - 2y^2) = 1$ is the required solution.

EXAMPLE 20 Find the particular solution of the differential equation

$$xy \frac{dy}{dx} = (x+2)(y+2), \text{ it being given that } y = -1 \text{ when } x = 1.$$

[CBSE 2012]

SOLUTION The given differential equation is

$$xy \frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \frac{y}{(y+2)} dy = \frac{(x+2)}{x} dx$$



$$\Rightarrow \int \frac{y}{(y+2)} dy = \int \frac{(x+2)}{x} dx + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\Rightarrow \int \left\{ 1 - \frac{2}{(y+2)} \right\} dy = \int \left(1 + \frac{2}{x} \right) dx + C$$

$$\Rightarrow y - 2 \log |y+2| = x + 2 \log |x| + C. \quad \dots (i)$$

Putting $x = 1$ and $y = -1$ in (i), we get $C = -2$.

Hence, the required solution is

$$y - 2 \log |y+2| = x + 2 \log |x| - 2.$$

EXAMPLE 21 Find the particular solution of the differential equation $x(x^2 - 1) \frac{dy}{dx} = 1$,

it being given that $y = 0$ when $x = 2$.

[CBSE 2012]

SOLUTION We have

$$x(x^2 - 1) \frac{dy}{dx} = 1$$

$$\Rightarrow dy = \frac{1}{x(x^2 - 1)} dx$$

$$\Rightarrow \int dy = \int \frac{dx}{x(x-1)(x+1)}. \quad \dots (i)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+1)}.$$

$$\text{Then, } A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \equiv 1. \quad \dots (ii)$$

Putting $x = 0$ in (ii), we get $A = -1$.

Putting $x = 1$ in (ii), we get $B = \frac{1}{2}$.

Putting $x = -1$ in (ii), we get $C = \frac{1}{2}$.

$$\therefore \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}.$$

Putting this value in (i), we get

$$\int dy = \int \frac{-dx}{x} + \frac{1}{2} \int \frac{dx}{(x-1)} + \frac{1}{2} \int \frac{dx}{(x+1)} + C_1,$$

where C_1 is an arbitrary constant

$$\Rightarrow y = -\log |x| + \frac{1}{2} \log |x-1| + \frac{1}{2} \log |x+1| + C_1. \quad \dots (iii)$$

We are given that $y = 0$ when $x = 2$.

Putting $x = 2$ and $y = 0$ in (iii), we get

$$-\log 2 + \frac{1}{2} \log 3 + C_1 = 0$$

$$\Rightarrow C_1 = \log 2 - \frac{1}{2} \log 3 = \frac{1}{2} \log 4 - \frac{1}{2} \log 3 = \frac{1}{2} \log \left(\frac{4}{3} \right).$$



Putting $C_1 = \frac{1}{2} \log \left(\frac{4}{3} \right)$ in (iii) we get

$$\begin{aligned} y &= -\log |x| + \frac{1}{2} \log |x-1| + \frac{1}{2} \log |x+1| + \frac{1}{2} \log \frac{4}{3} \\ \Rightarrow y &= -\frac{1}{2} \log x^2 + \frac{1}{2} \log |(x-1)(x+1)| + \frac{1}{2} \log \frac{4}{3} \\ \Rightarrow y &= \frac{1}{2} \log \left| \frac{4(x^2-1)}{3x^2} \right|, \text{ which is the required solution.} \end{aligned}$$

EXAMPLE 22 Find the particular solution of the differential equation $(x+1) \frac{dy}{dx} = 2e^{-y} - 1$, it being given that $y = 0$ when $x = 0$. [CBSE 2012]

SOLUTION We have

$$\begin{aligned} (x+1) \frac{dy}{dx} &= 2e^{-y} - 1 \\ \Rightarrow \frac{1}{(2e^{-y} - 1)} dy &= \frac{1}{(x+1)} dx \\ \Rightarrow \frac{e^y}{(2 - e^y)} dy &= \frac{1}{(x+1)} dx \\ \Rightarrow \int \frac{e^y}{(e^y - 2)} dy &= \int \frac{-1}{(x+1)} dx \\ \Rightarrow \log |e^y - 2| &= -\log |x+1| + \log |C_1|, \\ &\text{where } C_1 \text{ is an arbitrary constant} \\ \Rightarrow \log |e^y - 2| + \log |x+1| &= \log |C_1| \\ \Rightarrow \log |(x+1)(e^y - 2)| &= \log |C_1| \\ \Rightarrow (x+1)(e^y - 2) &= \pm C_1 = C. \quad \dots (i) \end{aligned}$$

It is given that when $x = 0$, then $y = 0$.

Putting $x = 0$ and $y = 0$ in (i), we get $C = -1$.

Putting $C = -1$ in (i), we get

$$\begin{aligned} (x+1)(e^y - 2) &= -1 \Rightarrow (e^y - 2) = \frac{-1}{(x+1)} \\ \Rightarrow e^y &= \left\{ 2 - \frac{1}{(x+1)} \right\} = \frac{(2x+1)}{(x+1)}. \\ \therefore y &= \log \left| \frac{2x+1}{x+1} \right|, x \neq -1 \text{ is the required solution.} \end{aligned}$$

EXAMPLE 23 Solve the differential equation

$$(1+y^2)(1+\log x) dx + x dy = 0, \text{ it being given that } y = 1 \text{ when } x = 1. \quad [\text{CBSE 2000, '03, '11}]$$

SOLUTION We have

$$(1+y^2)(1+\log x) dx + x dy = 0$$



$$\begin{aligned} \Rightarrow \frac{(1 + \log x)}{x} dx + \frac{1}{(1 + y^2)} dy &= 0 \quad [\text{on separating the variables}] \\ \Rightarrow \int \frac{(1 + \log x)}{x} dx + \int \frac{1}{(1 + y^2)} dy &= C \quad [\text{integrating both sides}] \\ \Rightarrow \int t dt + \tan^{-1} y &= C, \text{ where } (1 + \log x) = t \\ \Rightarrow \frac{1}{2} t^2 + \tan^{-1} y &= C, \text{ where } C \text{ is an arbitrary constant} \\ \Rightarrow \frac{1}{2} (1 + \log x)^2 + \tan^{-1} y &= C. \quad \dots (i) \end{aligned}$$

Putting $x = 1$ and $y = 1$ in (i), we get

$$C = \frac{1}{2} + \tan^{-1} 1 \Rightarrow C = \left(\frac{1}{2} + \frac{\pi}{4} \right). \quad \dots (ii)$$

$$\therefore \frac{1}{2} (1 + \log x)^2 + \tan^{-1} y = \left(\frac{1}{2} + \frac{\pi}{4} \right) \quad [\text{using (ii) in (i)}]$$

$$\Rightarrow \frac{1}{2} (\log x)^2 + \log x + \tan^{-1} y = \frac{\pi}{4}, \text{ which is the required solution.}$$

EXAMPLE 24 Solve the differential equation

$$(1 + e^{2x}) dy + e^x (1 + y^2) dx = 0,$$

it being given that $y = 1$ when $x = 0$.

[CBSE 2004, '08C, '11]

SOLUTION We have

$$\begin{aligned} (1 + e^{2x}) dy + e^x (1 + y^2) dx &= 0 \\ \Rightarrow \frac{1}{(1 + y^2)} dy + \frac{e^x}{(1 + e^{2x})} dx &= 0 \quad [\text{separating the variables}] \\ \Rightarrow \int \frac{1}{(1 + y^2)} dy + \int \frac{e^x}{(1 + e^{2x})} dx &= C \quad [\text{integrating both sides}] \\ \Rightarrow \tan^{-1} y + \int \frac{dt}{(1 + t^2)} &= C, \text{ where } e^x = t \\ \Rightarrow \tan^{-1} y + \tan^{-1} t &= C \\ \Rightarrow \tan^{-1} y + \tan^{-1} e^x &= C \quad \dots (i) \quad [\because t = e^x] \end{aligned}$$

Putting $x = 0$ and $y = 1$ in (i), we get

$$C = \tan^{-1} 1 + \tan^{-1} e^0 = (\tan^{-1} 1 + \tan^{-1} 1) = \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\pi}{2}.$$

$$\therefore \tan^{-1} y + \tan^{-1} e^x = \frac{\pi}{2} \text{ is the required solution.}$$

EXAMPLE 25 Find the equation of the curve that passes through the point $(1, 2)$ and satisfies the differential equation $\frac{dy}{dx} = \frac{-2xy}{(x^2 + 1)}$.

SOLUTION We have

$$\frac{dy}{dx} = \frac{-2xy}{(x^2 + 1)}$$



$$\Rightarrow \frac{dy}{y} = \frac{-2x}{(x^2 + 1)} dx \quad [\text{on separating the variables}]$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{-2x}{(x^2 + 1)} dx \quad [\text{integrating both sides}]$$

$$\Rightarrow \log y = -\log (x^2 + 1) + \log C,$$

where $\log C$ is an arbitrary constant

$$\Rightarrow \log y + \log (x^2 + 1) = \log C$$

$$\Rightarrow \log \{y(x^2 + 1)\} = \log C$$

$$\Rightarrow y(x^2 + 1) = C \quad \dots (i)$$

Now, it is given that the curve passes through (1, 2).

So, putting $x = 1$ and $y = 2$ in (i), we get $C = 4$.

$\therefore y(x^2 + 1) = 4$ is the required equation of the curve.

EXAMPLE 26 Find the equation of a curve which passes through the point $(-2, 3)$ and the slope of whose tangent at any point (x, y) is $\frac{2x}{y^2}$.

SOLUTION We know that the slope of a curve at a point (x, y) is $\frac{dy}{dx}$.

$$\therefore \frac{dy}{dx} = \frac{2x}{y^2} \quad \dots (i)$$

$$\Rightarrow y^2 dy = 2x dx \quad [\text{separating the variables}]$$

$$\Rightarrow \int y^2 dy = \int 2x dx$$

$$\Rightarrow \frac{1}{3} y^3 = x^2 + C \quad \dots (ii)$$

where C is a constant.

Thus, (ii) is the equation of the curve whose differential equation is given by (i).

Since the given curve passes through the point $(-2, 3)$, we have

$$C = \left(\frac{1}{3} \times 27 \right) - (-2)^2 = (9 - 4) = 5.$$

Hence, the required equation of the curve is

$$\frac{1}{3} y^3 = x^2 + 5 \Rightarrow y^3 = 3x^2 + 15.$$

EXAMPLE 27 In a bank principal increases at the rate of 5% per annum. In how many years will ₹ 1000 double itself?

SOLUTION Let P be the principal at any time t . Then,

$$\frac{dP}{dt} = \left(\frac{5}{100} \right) P$$

$$\Rightarrow \frac{dP}{dt} = \frac{P}{20}$$

$$\Rightarrow \frac{dP}{P} = \frac{1}{20} dt$$



$$\begin{aligned} \Rightarrow \int \frac{dP}{P} &= \int \frac{1}{20} dt \quad [\text{on integrating both sides}] \\ \Rightarrow \log P &= \frac{1}{20}t + \log C, \text{ where } C \text{ is an arbitrary constant} \\ \Rightarrow \log \left(\frac{P}{C} \right) &= \frac{t}{20} \\ \Rightarrow \frac{P}{C} &= e^{t/20}. \end{aligned} \quad \dots (i)$$

When $t = 0$, we have $P = 1000$ (given).

Putting $P = 1000$ and $t = 0$ in (i), we get $C = 1000$.

$$\therefore P = (1000)e^{t/20}. \quad \dots (ii)$$

Let ₹ 1000 double itself in n years.

Thus, when $t = n$, then $P = 2000$.

Putting these values in (ii), we get

$$\begin{aligned} (1000) \times e^{n/20} &= 2000 \Rightarrow e^{n/20} = 2 \\ &\Rightarrow \frac{n}{20} = \log_e 2 \Rightarrow n = 20(\log_e 2). \end{aligned}$$

Hence, the required time is $\{20(\log_e 2)\}$ years.

EXERCISE 19A

Very-Short-Answer Questions

Find the general solution of each of the following differential equations:

1. $\frac{dy}{dx} = (1+x^2)(1+y^2)$
2. $x^4 \frac{dy}{dx} = -y^4$
3. $\frac{dy}{dx} = 1+x+y+xy$ [CBSE 2012]
4. $\frac{dy}{dx} = 1-x+y-xy$
5. $(x-1)\frac{dy}{dx} = 2x^3y$
6. $\frac{dy}{dx} = e^{x+y}$
7. $(e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$
8. $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ [CBSE 2006]
9. $e^{2x-3y}dx + e^{2y-3x}dy = 0$
10. $e^x \tan y dx + (1-e^x) \sec^2 y dy = 0$ [CBSE 2011]
11. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$
12. $\cos x(1+\cos y)dx - \sin y(1+\sin x)dy = 0$

For each of the following differential equations, find a particular solution satisfying the given condition:

13. $\cos \left(\frac{dy}{dx} \right) = a$, where $a \in \mathbb{R}$ and $y = 2$ when $x = 0$.
14. $\frac{dy}{dx} = -4xy^2$, it being given that $y = 1$ when $x = 0$.



15. $x dy = (2x^2 + 1)dx$ ($x \neq 0$), given that $y = 1$ when $x = 1$.

16. $\frac{dy}{dx} = y \tan x$, it being given that $y = 1$ when $x = 0$.

ANSWERS (EXERCISE 19A)

1. $\tan^{-1}y = x + \frac{x^3}{3} + C$ 2. $\frac{1}{x^3} + \frac{1}{y^3} = C$ 3. $\log |1 + y| = x + \frac{x^2}{2} + C$

4. $\log |1 + y| = x - \frac{x^2}{2} + C$ 5. $\log |y| = \frac{2x^3}{3} + x^2 + 2x + 2\log |x - 1| + C$

6. $e^x + e^{-y} = C$ 7. $y = \log(e^x + e^{-x}) + C$ 8. $e^y = e^x + \frac{x^3}{3} + C$

9. $e^{5x} + e^{5y} = C$ 10. $\tan y = C(1 - e^x)$ 11. $\tan x \tan y = C$

12. $(1 + \sin x)(1 + \cos y) = C$ 13. $\cos\left(\frac{y-2}{x}\right) = a$ 14. $y = \frac{1}{(2x^2 + 1)}$

15. $y = x^2 + \log |x|$ 16. $y = \sec x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 19A)

1. $\int \frac{1}{(1 + y^2)} dy = \int (1 + x^2) dx + C.$

2. $\frac{-1}{y^4} dy = \frac{1}{x^4} dx \Rightarrow \int x^{-4} dx + \int y^{-4} dy = C_1 \Rightarrow \frac{1}{x^3} + \frac{1}{y^3} = -3C_1 = C.$

3. $\frac{dy}{dx} = (1 + x) + y(1 + x) = (1 + x)(1 + y) \Rightarrow \int \frac{dy}{(1 + y)} = \int (1 + x) dx + C.$

5. $\frac{1}{y} dy = \frac{2x^3}{(x-1)} = 2 \left[x^2 + x + 1 + \frac{1}{(x-1)} \right]$ [on dividing x^3 by $(x-1)$]

6. $\frac{dy}{dx} = e^x \cdot e^y \Rightarrow \frac{1}{e^y} dy = e^x dx \Rightarrow \int e^{-y} dy = \int e^x dx + C$

7. $dy - \frac{(e^x - e^{-x})}{(e^x + e^{-x})} dx = 0 \Rightarrow \int dy - \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx + C \Rightarrow y - \log(e^x + e^{-x}) = C.$

8. $\frac{dy}{dx} = e^x \cdot e^{-y} + x^2 e^{-y} = (e^x + x^2) e^{-y} \Rightarrow \int e^y dy = \int (e^x + x^2) dx + C.$

9. $e^{2x} \cdot e^{-3y} dx + e^{2y} \cdot e^{-3x} dy = 0 \Rightarrow \int e^{5x} dx + \int e^{5y} dy = C.$

10. $-\int \frac{-e^x}{(1 - e^x)} dx + \int \frac{\sec^2 y}{\tan y} dy = \log |C_1| \Rightarrow \log |\tan y| - \log |1 - e^x| = \log |C_1|.$

$\therefore \log \left| \frac{\tan y}{(1 - e^x)} \right| = \log |C_1| \Rightarrow \frac{\tan y}{(1 - e^x)} = \pm C_1 = C.$



11. $\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = \log |C_1| \Rightarrow \log |\tan x| + \log |\tan y| = \log |C_1|$
 $\therefore \log |\tan x \tan y| = \log |C_1| \Rightarrow |\tan x \tan y| = |C_1|$
 $\therefore \tan x \tan y = \pm C_1 = C$.
12. $\int \frac{\cos x}{(1 + \sin x)} dx + \int \frac{-\sin y}{(1 + \cos y)} dy = \log |C_1|$
 $\Rightarrow \log |1 + \sin x| + \log |1 + \cos y| = \log |C_1| \Rightarrow \log |(1 + \sin x)(1 + \cos y)| = \log |C_1|$
 $\Rightarrow |(1 + \sin x)(1 + \cos y)| = C_1 \Rightarrow (1 + \sin x)(1 + \cos y) = \pm C_1 = C$.
13. $\frac{dy}{dx} = \cos^{-1} a \Rightarrow \int dy = \int (\cos^{-1} a) dx \Rightarrow y = (\cos^{-1} a)x + C$ (i)
 Putting $x = 0$ and $y = 2$ in (i), we get $C = 2$.
 $\therefore \cos^{-1} a = \left(\frac{y-2}{x} \right) \Rightarrow \cos \left(\frac{y-2}{x} \right) = a$.
14. $\int \frac{dy}{y^2} = \int -4x dx \Rightarrow \frac{-1}{y} = -2x^2 + C \Rightarrow y = \frac{1}{(2x^2 - C)}$ (i)
 Putting $x = 0$ and $y = 1$ in (i), we get $C = -1$.
 Hence, $y = \frac{1}{(2x^2 + 1)}$.
15. $\int dy = \int \frac{(2x^2 + 1)}{x} dx = \int \left(2x + \frac{1}{x} \right) dx \Rightarrow y = x^2 + \log |x| + C$ (i)
 Putting $x = 1$ and $y = 1$ in (i), we get $C = 0$. Hence, $y = x^2 + \log |x|$.
16. $\int \frac{1}{y} dy = \int \tan x dx \Rightarrow \log |y| = \log |\sec x| + \log |C_1|$
 $\therefore (y \cos x) = \pm C_1 = C$.
 When $x = 0$ and $y = 1$, we get $C = 1$. Hence, $y = \sec x$.

EXERCISE 19B

Find the general solution of each of the following differential equations:

- $\frac{dy}{dx} = \frac{x-1}{y+2}$
- $\frac{dy}{dx} = \frac{x}{(x^2+1)}$
- $\frac{dy}{dx} = (1+x)(1+y^2)$
- $(1+x^2) \frac{dy}{dx} = xy$
- $\frac{dy}{dx} + y = 1 \ (y \neq 1)$
- $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$
- $x \frac{dy}{dx} + y = y^2$
- $x^2(y+1)dx + y^2(x-1)dy = 0$
- $y(1-x^2) \frac{dy}{dx} = x(1+y^2)$
- $y \log y dx - x dy = 0$
- $x(x^2 - x^2 y^2) dy + y(y^2 + x^2 y^2) dx = 0$
- $(1-x^2) dy + xy(1-y) dx = 0$



$$13. (1-x^2)(1-y)dx = xy(1+y)dy$$

$$14. (y+xy)dx + (x-xy^2)dy = 0$$

[CBSE 2002C]

$$15. (x^2-yx^2)dy + (y^2+xy^2)dx = 0$$

[CBSE 2004C]

$$16. (x^2y-x^2)dx + (xy^2-y^2)dy = 0$$

$$17. x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$$

$$18. \frac{dy}{dx} = e^{x+y} + x^2e^y$$

[CBSE 2006]

$$19. \frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$$

$$20. 3e^x \tan y dx + (1-e^x) \sec^2 y dy = 0$$

$$21. e^y(1+x^2)dy - \frac{x}{y}dx = 0$$

$$22. \frac{dy}{dx} = e^{x+y} + e^{x-y}$$

$$23. (e^y + 1) \cos x dx + e^y \sin x dy = 0$$

$$24. \frac{dy}{dx} + \frac{xy+y}{xy+x} = 0$$

$$25. \sqrt{1-x^4}dy = x dx$$

$$26. \operatorname{cosec} x \log y \frac{dy}{dx} + x^2y = 0$$

$$27. y dx + (1+x^2) \tan^{-1} x dy = 0$$

$$28. \frac{1}{x} \cdot \frac{dy}{dx} = \tan^{-1} x$$

$$29. e^x \sqrt{1-y^2} dx + \frac{y}{x} dy = 0$$

$$30. \frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$$

$$31. (\cos x) \frac{dy}{dx} + \cos 2x = \cos 3x$$

$$32. \frac{dy}{dx} + \frac{(1+\cos 2y)}{(1-\cos 2x)} = 0$$

$$33. \frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$$

$$34. \cos x(1+\cos y)dx - \sin y(1+\sin x)dy = 0$$

$$35. \sin^3 x dx - \sin y dy = 0$$

$$36. \frac{dy}{dx} + \sin(x+y) = \sin(x-y)$$

$$37. \frac{1}{x} \cos^2 y dy + \frac{1}{y} \cos^2 x dx = 0$$

$$38. \frac{dy}{dx} = \sin^3 x \cos^2 x + xe^x$$

$$39. \text{Find the particular solution of the differential equation } \frac{dy}{dx} = 1 + x + y + xy, \text{ given that } y = 0 \text{ when } x = 1.$$

[CBSE 2014]

$$40. \text{Find the particular solution of the differential equation } x(1+y^2)dx - y(1+x^2)dy = 0, \text{ given that } y = 1 \text{ when } x = 0.$$

[CBSE 2014]



41. Find the particular solution of the differential equation $\log \left(\frac{dy}{dx} \right) = 3x + 4y$, given that $y = 0$ when $x = 0$. [CBSE 2014]
42. Solve the differential equation $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$, given that $y = 1$ when $x = 1$. [CBSE 2014]
43. Find the particular solution of the differential equation $e^x \sqrt{1 - y^2} dx + \frac{y}{x} dy = 0$, given that $y = 1$ when $x = 0$. [CBSE 2014]
44. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{(\sin y + y \cos y)}$, given that $y = \frac{\pi}{2}$ when $x = 1$. [CBSE 2014]
45. Solve the differential equation $\frac{dy}{dx} = y \sin 2x$, given that $y(0) = 1$. [CBSE 2004]
46. Solve the differential equation $(x + 1) \frac{dy}{dx} = 2xy$, given that $y(2) = 3$. [CBSE 2004]
47. Solve $\frac{dy}{dx} = x(2\log x + 1)$, given that $y = 0$ when $x = 2$.
48. Solve $(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x$, given that $y = 1$ when $x = 0$.
49. Solve $\frac{dy}{dx} = y \tan x$, given that $y = 1$ when $x = 0$.
50. Solve $\frac{dy}{dx} = y^2 \tan 2x$, given that $y = 2$ when $x = 0$.
51. Solve $\frac{dy}{dx} = y \cot 2x$, given that $y = 2$ when $x = \frac{\pi}{4}$.
52. Solve $(1 + x^2) \sec^2 y dy + 2x \tan y dx = 0$, given that $y = \frac{\pi}{4}$ when $x = 1$.
53. Find the equation of the curve passing through the point $\left(0, \frac{\pi}{4}\right)$ whose differential equation is $\sin x \cos y dx + \cos x \sin y dy = 0$.
54. Find the equation of a curve which passes through the origin and whose differential equation is $\frac{dy}{dx} = e^x \sin x$.
55. A curve passes through the point $(0, -2)$ and at any point (x, y) of the curve, the product of the slope of its tangent and y -coordinate of the point is equal to the x -coordinate of the point. Find the equation of the curve.
56. A curve passes through the point $(-2, 1)$ and at any point (x, y) of the curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve.



57. In a bank, principal increases at the rate of $r\%$ per annum. Find the value of r if ₹ 100 double itself in 10 years. (Given $\log_e 2 = 0.6931$)
58. In a bank, principal increases at the rate of 5% per annum. An amount of ₹ 1000 is deposited in the bank. How much will it worth after 10 years? (Given $e^{0.5} = 1.648$)
59. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.
60. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?

ANSWERS (EXERCISE 19B)

1. $y^2 + 4y - x^2 + 2x = C$ 2. $y = \frac{1}{2} \log (x^2 + 1) + C$ 3. $\tan^{-1} y = x + \frac{x^2}{2} + C$
4. $y = C_1 \sqrt{1 + x^2}$ 5. $x + \log |1 - y| = C$ 6. $\sin^{-1} y + \sin^{-1} x = C$
7. $y = 1 + C_1 xy$ 8. $\frac{x^2}{2} + \frac{y^2}{2} + x - y + \log |x - 1| + \log |y + 1| = C$
9. $(1 + y^2)(1 - x^2) = C$ 10. $x = C \log y$ 11. $\frac{-1}{2y^2} - \frac{1}{2x^2} + \log \left| \frac{x}{y} \right| = C$
12. $y = C(1 - y)\sqrt{1 - x^2}$ 13. $\log |x(1 - y^2)| = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$
14. $\log |xy| + x - \frac{y^2}{2} = C$ 15. $\log \left| \frac{x}{y} \right| = \frac{1}{x} + \frac{1}{y} + C$
16. $\frac{1}{2}(x^2 + y^2) + (x + y) + \log |(x - 1)(y - 1)| = C$ 17. $\sqrt{1 + y^2} + \sqrt{1 + x^2} = C$
18. $e^x + e^{-y} + \frac{x^3}{3} = C$ 19. $y = e^{3x} + C$ 20. $\tan y = C(1 - e^{-x})^3$
21. $(y - 1)e^y = \frac{1}{2} \log (1 + x^2) + C$ 22. $\tan^{-1}(e^y) = e^x + C$
23. $(1 + e^y) \sin x = C$ 24. $x + y + \log |xy| = C$ 25. $y = \frac{1}{2} \sin^{-1}(x^2) + C$
26. $\frac{1}{2}(\log y)^2 + (2 - x^2) \cos x + 2x \sin x = C$ 27. $y \tan^{-1} x = C$
28. $y = \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2}x + C$ 29. $(x - 1)e^x - \sqrt{1 - y^2} = C$
30. $y = 2 \tan \frac{x}{2} - x + C$ 31. $y = \sin 2x - 2 \sin x - x + \log |\sec x + \tan x| + C$
32. $\tan y = \cot x + C$ 33. $\log |\sin y| + \sin x = C$



34. $(1 + \sin x)(1 + \cos y) = C$ 35. $12 \cos y = 9 \cos x - \cos 3x + C$
 36. $\log |\operatorname{cosec} y - \cot y| + 2 \sin x = C$
 37. $2(x^2 + y^2) + 2x \sin 2x + 2y \sin 2y + \cos 2x + \cos 2y = C$
 38. $y = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + xe^x - e^x + C$ 39. $\log |1 + y| = x + \frac{x^2}{2} - \frac{3}{2}$
 40. $y = \sqrt{2x^2 + 1}$ 41. $4e^{3x} + 3e^{-4y} = 7$ 42. $\log |y| + \frac{1}{y} + \frac{1}{x} - x = 1$
 43. $e^x(x - 1) = \sqrt{1 - y^2} - 1$ 44. $y \sin y = x^2 \log x + \frac{\pi}{2}$ 45. $y = e^{\sin^2 x}$
 46. $y(x + 1)^2 = 27e^{(2x-4)}$ 47. $y = x^2 \log x - 4 \log 2$
 48. $y = \frac{1}{2} \left\{ \log |x + 1| + \frac{3}{2} \log (x^2 + 1) - \tan^{-1} x \right\} + 1$ 49. $y \cos x = 1$
 50. $y(1 + \log |\cos 2x|) = 2$ 51. $y = 2\sqrt{\sin 2x}$ 52. $(1 + x^2) \tan y = 2$
 53. $y = \cos^{-1} \left(\frac{1}{\sqrt{2}} \sec x \right)$ 54. $2y = e^x (\sin x - \cos x) + 1$ 55. $y^2 = x^2 + 4$
 56. $y + 3 = (x + 4)^2$ 57. $r = 6.931$ 58. ₹ 1648 59. $r = (63t + 27)^{1/3}$
 60. $\frac{2 \log 2}{\log \left(\frac{11}{10} \right)}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 19B)

4. $\int \frac{1}{y} dy = \frac{1}{2} \int \frac{2x}{(1 + x^2)} dx \Rightarrow \log |y| - \frac{1}{2} \log |1 + x^2| = \log |C|$
 $\Rightarrow \log \left| \frac{y}{\sqrt{1 + x^2}} \right| = \log |C| \Rightarrow \frac{y}{\sqrt{1 + x^2}} = \pm C = C_1.$
 5. $\frac{1}{(1 - y)} dy = dx \Rightarrow - \int \frac{-dy}{(1 - y)} = \int dx$
 $\Rightarrow \int dx + \int \frac{-dy}{(1 - y)} = C \Rightarrow x + \log |1 - y| = C.$
 7. $\int \frac{1}{y(y - 1)} dy = \int \frac{1}{x} dx \Rightarrow \int \left\{ \frac{1}{(y - 1)} - \frac{1}{y} \right\} dy = \int \frac{1}{x} dx$ [by partial fractions]
 $\Rightarrow \log |y - 1| - \log |y| = \log |x| + \log |C|$
 $\Rightarrow (y - 1) = \pm Cxy \Rightarrow (y - 1) = C_1 xy \Rightarrow y = 1 + C_1 xy.$
 8. $\int \frac{x^2}{(x - 1)} dx + \int \frac{y^2}{(y + 1)} dy = C \Rightarrow \int \left(x + 1 + \frac{1}{x - 1} \right) dx + \int \left(y - 1 + \frac{1}{y + 1} \right) dy = C.$
 9. $\int \frac{2y}{(1 + y^2)} dy = - \int \frac{-2x}{(1 - x^2)} dx \Rightarrow \log |1 + y^2| + \log |1 - x^2| = \log |C_1|$
 $\Rightarrow (1 + y^2)(1 - x^2) = \pm C_1 = C.$



10. $\int \frac{1}{x} dx = \int \frac{1}{y \log y} dy \Rightarrow \int \frac{1}{x} dx = \int \frac{dt}{t}$, where $\log y = t$.
 $\therefore \log |x| = \log |t| + \log |C_1| \Rightarrow \log |x| - \log |\log y| = \log |C_1|$
 $\Rightarrow \log \left| \frac{x}{\log y} \right| = \log C_1 \Rightarrow \frac{x}{\log y} = \pm C_1 = C.$
11. $x^3(1-y^2)dy + y^3(1+x^2)dx = 0 \Rightarrow \int \frac{(1-y^2)}{y^3} dy + \int \frac{(1+x^2)}{x^3} dx = C.$
 $\therefore \int \left(y^{-3} - \frac{1}{y} \right) dy + \int \left(x^{-3} + \frac{1}{x} \right) dx = C \Rightarrow \frac{y^{-2}}{-2} - \log |y| + \frac{x^{-2}}{-2} + \log |x| = C.$
12. $\int \frac{1}{y(1-y)} dy + \int \frac{x}{(1-x^2)} dx = \log |C_1|$
 $\Rightarrow \int \left\{ \frac{1}{y} + \frac{1}{(1-y)} \right\} dy - \frac{1}{2} \int \frac{-2x}{(1-x^2)} dx = \log |C_1|$
 $\Rightarrow \log |y| - \log |1-y| - \frac{1}{2} \log |1-x^2| = \log |C_1| \Rightarrow \log \left| \frac{y}{(1-y)\sqrt{1-x^2}} \right| = \log |C_1|$
 $\Rightarrow \frac{y}{(1-y)\sqrt{1-x^2}} = \pm C_1 = C.$
13. $\int \frac{(1-x^2)}{x} dx = \int \frac{y(1+y)}{(1-y)} dy \Rightarrow \int \left(\frac{1}{x} - x \right) dx = \int \frac{(y^2+y)}{(-y+1)} dy$
 $\therefore \int \left(\frac{1}{x} - x \right) dx = \int \left(-y - 2 + \frac{2}{1-y} \right) dy$ [on dividing (y^2+y) by $(-y+1)$].
19. $\frac{dy}{dx} = \frac{3e^{2x}(1+e^{2x})}{\left(e^x + \frac{1}{e^x} \right)} = \frac{3e^{3x}(1+e^{2x})}{(1+e^{2x})} = 3e^{3x}.$
20. $\int \frac{3e^x}{(1-e^x)} dx + \int \frac{\sec^2 y}{\tan y} dy = \log |C_1|$
 $\Rightarrow -3 \int \frac{-e^x}{(1-e^x)} dx + \int \frac{\sec^2 y}{\tan y} dy = \log |C_1| \Rightarrow -3 \log |1-e^{-x}| + \log |\tan y| = \log |C_1|$
 $\Rightarrow \log |(1-e^{-x})^{-3} \tan y| = \log |C_1| \Rightarrow \frac{\tan y}{(1-e^{-x})^3} = \pm C_1 = C.$
22. $\int \frac{e^y}{(e^{2y}+1)} dy = \int e^x dx \Rightarrow \int \frac{dt}{(t^2+1)} = e^x + C$, where $e^y = t$
 $\Rightarrow \tan^{-1}(t) = e^x + C \Rightarrow \tan^{-1}(e^y) = e^x + C.$
26. $\int \frac{\log y}{y} dy + \int x^2 \sin x dx = C$
 $\Rightarrow \int t dt + x^2(-\cos x) - \int 2x(-\cos x) dx = C$, where $\log y = t$
 $\Rightarrow \frac{t^2}{2} - x^2 \cos x + 2 \int x \cos x dx = C.$
27. $\int \frac{dx}{(1+x^2) \tan^{-1} x} + \int \frac{1}{y} dy = \log |C_1|.$
Put $\tan^{-1} x = t$ and $\frac{dx}{(1+x^2)} = dt.$



$$28. \int dy = \int (\tan^{-1} x) x dx \Rightarrow y = (\tan^{-1} x) \frac{x^2}{2} - \int \frac{1}{(1+x^2)} \cdot \frac{x^2}{2} dx + C$$

$$\Rightarrow y = \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} \int \left\{ \frac{(1+x^2)-1}{(1+x^2)} \right\} dx + C$$

$$\Rightarrow y = \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} \int \left\{ 1 - \frac{1}{(1+x^2)} \right\} dx + C.$$

$$30. \frac{dy}{dx} = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2 \frac{x}{2} = \left(\sec^2 \frac{x}{2} - 1 \right)$$

$$\therefore \int dy = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx.$$

$$31. \frac{dy}{dx} = \frac{\cos 3x - \cos 2x}{\cos x} = \frac{(4\cos^3 x - 3\cos x) - (2\cos^2 x - 1)}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = 4\cos^2 x - 2\cos x - 3 + \sec x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4(1+\cos 2x)}{2} - 2\cos x - 3 + \sec x = 2\cos 2x - 2\cos x - 1 + \sec x.$$

$$32. \frac{dy}{dx} + \frac{2\cos^2 y}{2\sin^2 x} = 0 \Rightarrow \frac{dy}{dx} + \frac{\operatorname{cosec}^2 x}{\sec^2 y} = 0 \Rightarrow \int (\sec^2 y) dy + \int \operatorname{cosec}^2 x dx = C.$$

$$33. \int \cot y dy + \int \cos x dx = C.$$

$$34. \int \frac{\cos x}{(1+\sin x)} dx - \int \frac{\sin y}{(1+\cos y)} dy = \log |C_1|$$

$$\Rightarrow \log |1+\sin x| + \log |1+\cos y| = \log |C_1|$$

$$\Rightarrow \log |(1+\sin x)(1+\cos y)| = \log |C_1| \Rightarrow (1+\sin x)(1+\cos y) = \pm C_1 = C.$$

$$35. \sin y dy = \sin^3 x dx = \frac{(3\sin x - \sin 3x)}{4} dx \quad [\because \sin 3x = 3\sin x - 4\sin^3 x]$$

$$\therefore \int \sin y dy = \frac{3}{4} \int \sin x dx - \frac{1}{4} \int \sin 3x dx + C.$$

$$36. \sin(x-y) - \sin(x+y) = 2\cos\left\{\frac{(x-y)+(x+y)}{2}\right\} \sin\left\{\frac{(x-y)-(x+y)}{2}\right\}$$

$$= 2\cos x \sin(-y) = -2\cos x \sin y.$$

$$\therefore \int \frac{1}{\sin y} dy = -2 \int \cos x dx \Rightarrow \int \operatorname{cosec} y dy = -2\sin x + C$$

$$\Rightarrow \log |\operatorname{cosec} y - \cot y| = -2\sin x + C.$$

$$37. y \cos^2 y dy + x \cos^2 x dx = 0 \Rightarrow y(2\cos^2 y) dy + x(2\cos^2 x) dx = 0$$

$$\Rightarrow y(1+\cos 2y) dy + x(1+\cos 2x) dx = 0$$

$$\Rightarrow \int y dy + \int y \cos 2y dy + \int x dx + \int x \cos 2x dx = C$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \int_I y \cos 2y dy + \int_{II} x \cos 2x dx = C.$$

$$38. \int dy = \int \cos^2 x (1 - \cos^2 x) \sin x dx + \int x e^x dx + C$$

$$\Rightarrow y = -\int t^2 (1-t^2) dt + \int x e^x dx + C, \text{ where } t = \cos x$$

$$\Rightarrow y = -\int t^2 dt + \int t^4 dt + \int_I x e^x dx + C.$$



$$39. \frac{dy}{dx} = (1+x)(1+y) \Rightarrow \int \frac{dy}{(1+y)} = \int (1+x) dx \Rightarrow \log |1+y| = x + \frac{x^2}{2} + C. \quad \dots (i)$$

Putting $x = 1$ and $y = 0$ in (i), we get $C = \frac{-3}{2}$.

$$\therefore \log |1+y| = x + \frac{x^2}{2} - \frac{3}{2}.$$

$$40. \frac{1}{2} \int \frac{2x}{(1+x^2)} dx - \frac{1}{2} \int \frac{2y}{(1+y^2)} dy = \log |C_1|$$

$$\Rightarrow \frac{1}{2} \log (1+x^2) - \frac{1}{2} \log (1+y^2) = \log |C_1|$$

$$\Rightarrow \log (1+x^2) - \log (1+y^2) = \log |C_1^2| = \log C \Rightarrow \frac{(1+x^2)}{(1+y^2)} = C. \quad \dots (i)$$

Putting $x = 0$ and $y = 1$ in (i), we get $C = \frac{1}{2}$.

$$\therefore (1+y^2) = 2(1+x^2) \Rightarrow y = \sqrt{2x^2 + 1}.$$

$$41. \frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y} \Rightarrow \int e^{3x} dx = \int e^{-4y} dy$$

$$\therefore \frac{e^{3x}}{3} = \frac{e^{-4y}}{-4} + C. \quad \dots (i)$$

Putting $x = 0$ and $y = 0$ in (i), we get $C = \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7}{12}$.

$$\therefore \frac{e^{3x}}{3} = \frac{e^{-4y}}{-4} + \frac{7}{12} \Rightarrow 4e^{3x} + 3e^{-4y} = 7.$$

$$42. x^2(1-y)dy + y^2(1+x^2)dx = 0 \Rightarrow \int \frac{(1-y)}{y^2} dy + \int \left(\frac{1+x^2}{x^2} \right) dx = C_1.$$

$$\therefore -\frac{1}{y} - \log |y| - \frac{1}{x} + x = C_1 \Rightarrow \log |y| + \frac{1}{y} + \frac{1}{x} - x = C. \quad \dots (i)$$

Putting $x = 1$ and $y = 1$ in (i), we get $C = 1$.

$$43. \int_I x e^x dx = \frac{1}{2} \int \frac{-2y}{\sqrt{1-y^2}} dy \Rightarrow e^x(x-1) = \sqrt{1-y^2} + C. \quad \dots (i)$$

Putting $x = 0$ and $y = 1$ in (i), we get $C = -1$.

$$44. \int (\sin y + y \cos y) dy = \int (2x \log x + x) dx + C$$

$$\Rightarrow \int \sin y dy + \int y \cos y dy = \int 2x \log x dx + \int x dx + C$$

$$\Rightarrow (-\cos y) + y \sin y - \int 1 \cdot \sin y dy = 2 \left[(\log x) \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right] + \frac{x^2}{2} + C$$

$$\Rightarrow y \sin y = x^2 \log x + C. \quad \dots (i)$$

$$45. \int \frac{1}{y} dy = \int \sin 2x dx + C \Rightarrow \log |y| = -\frac{1}{2} \cos 2x + C. \quad \dots (i)$$

Putting $x = 0$ and $y = 1$ in (i), we get $C = \frac{1}{2}$.

$$\therefore \log |y| = \frac{1}{2} (1 - \cos 2x) = \left(\frac{1}{2} \times 2 \sin^2 x \right) = \sin^2 x.$$

Hence, $y = e^{\sin^2 x}$.



$$46. \int \frac{1}{y} dy = \int \frac{2x}{(x+1)} dx = 2 \int \frac{(x+1)-1}{(x+1)} dx = 2 \int \left\{ 1 - \frac{1}{(x+1)} \right\} dx$$

$$\Rightarrow \log |y| = 2x - 2 \log |x+1| + \log |C_1|$$

$$\Rightarrow \log |y| = \log |e^{2x}| - \log |(x+1)^2| + \log |C_1|$$

$$\Rightarrow \log \left| \frac{y(x+1)^2}{e^{2x}} \right| = \log |C_1| \Rightarrow \frac{y(x+1)^2}{e^{2x}} = \pm C_1 = C \text{ (say).}$$

$$\text{Then, } y(x+1)^2 = Ce^{2x}.$$

... (i)

$$\text{Putting } x = 2 \text{ and } y = 3 \text{ in (i), we get } C = 27e^{-4}.$$

$$\therefore y(x+1)^2 = 27e^{2x-4} \text{ is the required solution.}$$

$$48. dy = \frac{1}{2} \left\{ \frac{1}{(x+1)} + \frac{3x-1}{(x^2+1)} \right\} dx \quad [\text{by partial fractions}]$$

$$\Rightarrow \int dy = \frac{1}{2} \int \left\{ \frac{1}{(x+1)} + \frac{3}{2} \cdot \frac{2x}{(x^2+1)} - \frac{1}{(x^2+1)} \right\} dx + C$$

$$\Rightarrow y = \frac{1}{2} \left\{ \log |x+1| + \frac{3}{2} \log |x^2+1| - \tan^{-1} x \right\} + C.$$

$$\text{When } x = 0 \text{ and } y = 1, \text{ then } C = 1.$$

$$51. \int \frac{dy}{y} = \int \frac{\cos 2x}{\sin 2x} dx \Rightarrow \log |y| = \frac{1}{2} \log |\sin 2x| + \log |C_1|$$

$$\therefore \left| \frac{y}{\sqrt{\sin 2x}} \right| = C_1 \Rightarrow \frac{y}{\sqrt{\sin 2x}} = \pm C_1 = C \text{ (say).}$$

... (i)

$$\text{Putting } x = \frac{\pi}{4} \text{ and } y = 2 \text{ in (i), we get } C = 2.$$

$$\text{Hence, } y = 2\sqrt{\sin 2x}.$$

$$53. \tan x dx + \tan y dy = 0$$

$$\Rightarrow \int \tan x dx + \int \tan y dy = \text{constant}$$

$$\Rightarrow -\log |\cos x| - \log |\cos y| = \log |C_1|$$

$$\Rightarrow \log |\cos x \cos y| = \log \left| \frac{1}{C_1} \right| \Rightarrow \cos x \cos y = \pm \frac{1}{C_1} = C \text{ (say).}$$

... (i)

$$\text{Putting } x = 0 \text{ and } y = \frac{\pi}{4} \text{ in (i), we get } C = \frac{1}{\sqrt{2}}.$$

$$\therefore \cos x \cos y = \frac{1}{\sqrt{2}} \Rightarrow \cos y = \frac{1}{\sqrt{2}} \sec x \Rightarrow y = \cos^{-1} \left(\frac{1}{\sqrt{2}} \sec x \right).$$

$$54. \text{ Use the formula } \int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{(a^2 + b^2)}.$$

$$\text{Put } a = 1 \text{ and } b = 1.$$

$$55. y \frac{dy}{dx} = x \Rightarrow \int y dy = \int x dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C.$$

$$\text{Put } x = 0 \text{ and } y = -2 \text{ to get } C = 2.$$

$$56. \frac{dy}{dx} = \frac{2(y+3)}{(x+4)} \Rightarrow \int \frac{1}{(y+3)} dy = 2 \int \frac{1}{(x+4)} dx$$

$$\therefore \log |y+3| = 2 \log |x+4| + \log |C_1| \Rightarrow \frac{(y+3)}{(x+4)^2} = \pm C_1 = C.$$

$$\text{Putting } x = -2 \text{ and } y = 1, \text{ we get } C = 1.$$



$$57. \frac{dP}{dt} = \left(\frac{r}{100}\right)P \Rightarrow \int \frac{dP}{P} = \int \left(\frac{r}{100}\right) dt \Rightarrow \log P = \frac{rt}{100} + C. \quad \dots (i)$$

At $t = 0$, we have $P = P_0$. So, $\log P_0 = C$.

$$\therefore \log P - \log P_0 = \frac{rt}{100} \Rightarrow \log \left(\frac{P}{P_0}\right) = \frac{rt}{100}. \quad \dots (ii)$$

Putting $P_0 = 100$, $P = 2P_0 = 200$ and $t = 10$, we get

$$\frac{r}{10} = \log 2 \Rightarrow r = (10 \times 0.6931) = 6.931.$$

$$58. \frac{dP}{dt} = \frac{5P}{100} \Rightarrow \int \frac{dP}{P} = \int \frac{1}{20} dt \Rightarrow \log P = \frac{t}{20} + \log C.$$

At $t = 0$, we have $P = 1000$ and so $\log C = \log 1000$.

$$\therefore \log P = \frac{t}{20} + \log 1000.$$

$$\text{Putting } t = 10, \text{ we get } \log \left(\frac{P}{1000}\right) = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} = 1.648.$$

$$\therefore P = (1000 \times 1.648) = 1648.$$

$$59. \text{ The volume of a spherical balloon of radius } r \text{ is given by } V = \frac{4}{3}\pi r^3.$$

Now, $\frac{dV}{dt} = -k$, where $k > 0$ [note that V is decreasing]

$$\Rightarrow \frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right) = -k \Rightarrow (4\pi r^2) \frac{dr}{dt} = -k$$

$$\Rightarrow \int (4\pi r^2) dr = \int (-k) dt$$

$$\Rightarrow \frac{4}{3}\pi r^3 = -kt + C \quad \dots (i), \text{ where } C \text{ is an arbitrary constant.}$$

Putting $t = 0$ and $r = 3$ in (i), we get $C = 36\pi$.

$$\therefore \frac{4}{3}\pi r^3 = -kt + 36\pi. \quad \dots (ii)$$

It is being given that when $t = 3$, then $r = 6$.

Putting $t = 3$ and $r = 6$ in (ii), we get $k = -84\pi$.

Putting $k = -84\pi$ in (ii), we get

$$r^3 = (63t + 27) \Rightarrow r = (63t + 27)^{1/3}.$$

60. Let at any time t , the bacteria count be N . Then,

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN \Rightarrow \int \frac{1}{N} dN = \int k dt \Rightarrow \log N = kt + \log(C).$$

At $t = 0$, we have $N = 100000$.

$$\therefore \log C = \log 100000$$

$$\Rightarrow \log N = kt + \log 100000.$$

At $t = 2$, we have $N = 110000$.

$$\text{Putting these values in (i), we get } k = \frac{1}{2} \log \frac{11}{10}.$$

$$\therefore \log N = \frac{1}{2}t \log \left(\frac{11}{10}\right) + \log 100000. \quad \dots (ii)$$

When $N = 200000$, let $t = T$, then

$$\log 200000 = \frac{T}{2} \log \left(\frac{11}{10}\right) + \log 100000 \Rightarrow T = \frac{2 \log 2}{\log \left(\frac{11}{10}\right)}.$$

